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Holomorphic Anomaly Equation

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Abstract: ~~It's not Abstract~~

It's quite abstract so I didn't understand these stuff well, so please point out mistakes

Contents

0 Introduction

What makes Topological supersymmetric models become Topological string theory is gravity shows up, and what makes topological string theory simple and powerful is its correlation function is simple enough to calculate to arbitrary loops, up to a ambiguity.

Holomorphic Anomaly Equation(HAE) is the proper relation describing this process, we can even say that almost all calculable stringy effects of Topological string comes from HAE. We know properties like mirror symmetry or chiral ring is merely a supersymmetric effect, and after we extend our theory to string theory, many of them persists but some of them changes, those changes are coded in HAE.

This note is organized as follow: We first revisit the B model without gravity in Chap1, a very short remind of special geometry is in the appendix. We derive the HAE in Chap2. We then solve the HAE and make some comments of its properties in Chap3 and Chap4. We mainly follows BCOV's long paper (they have two), other references are in the bibliography part.

1 Twisted $\mathcal{N} = 2$ revisited

We shall quickly revisit the properties of $\mathcal{N} = 2$ theory, properties of tt^* equation,and couple it to topological gravity, where we will see why Calabi-Yau threefold is special and essential for our definition of B model.

We will use the notation of the original BCOV paper,which may have some difference with other references, but results are the same. It's well known that mirror symmetry can be manifested in $\mathcal{N} = 2$ (more specifically ,$\mathcal{N} = (2, 2)$), but in order to discover more property of mirror symmetry and made the theory a string theory, we consider the superconformal $\mathcal{N} = 2$ (specifically, $\mathcal{N} = (2, 2)$) now.

1.1 Vacuum geometry and twisting of $\mathcal{N} = 2$ theories

1.1.1 Topological twist and Chiral ring

Superconformal $\mathcal{N} = 2$ theories have four supercharges and two $U(1)$ currents, as the R symmetry

$$G^{\pm}, \bar{G}^{\pm}, J, \bar{J} \tag{1.1}$$

where we note left ones as the one without bar, right with bar ,satisfying these relation (left, but right are similar)

$$\begin{aligned}
&(G^{\pm})^2 = 0 \\
&\{G^+, G^-\} = 2H_L \\
&[G^{\pm}, H_L] = 0
\end{aligned} \tag{1.2}$$

and we have two equivalent choice of composed supercharge as cohomology operator, as the A and B model

$$Q_1 = G^+ + \bar{G}^+ \quad Q_2 = G^+ + \bar{G}^- \tag{1.3}$$

we can define the cohomology field theory

$$[Q, \phi] = 0 \quad \phi \sim \phi + [Q, \Lambda] \tag{1.4}$$

In our notation, satisfying this condition for Q_1 is named (c, c) and Q_2 is named (c, a), we should remember $Q_1^\dagger$ and $Q_2^\dagger$(named as (a, a),(a, c)) gives the isomorphic spectrum of fields, only differ by a $\bar{J} \to -\bar{J}$, so we can only consider Q_1 for simplicity. $\mathcal{N} = (2, 2)$ chiral primary operator has relation between of conformal weight and R-symmetry charges

$$(h_i, \bar{h}_i) = \frac{1}{2}(q_i, \bar{q}_i) \tag{1.5}$$

and the R symmetry charge is bounded by the central charge of superconformal algebra, we note as $\hat{c}$. For the algebraic relation of the chiral ring, we choose the normal notation

$$\phi_i \phi_j = C_{ij}^k \phi_k + [Q, \cdot] \tag{1.6}$$

one specific properties is by viewing the chiral field as the first component of a superfield, also named as the descend equation. We can modify the action by perturbing the action in these chiral fields

$$t^i \int d^2 z d^2 \theta^+ \phi_i + \bar{t}^i \int d^2 z d^2 \theta^- \bar{\phi}_i = t^i \int d^2 z \phi_i^{(2)} + \bar{t}^i \int d^2 z \bar{\phi}_i^{(2)} \tag{1.7}$$

where $\phi_i^{(2)} = \{G^-, [\bar{G}^-, \phi_i]\}$ are result of the descend equation.We recall:

$$\begin{aligned} &[Q, \mathcal{O}^{(0)} = 0 && \{Q, \mathcal{O}^{(1)}\} = d\mathcal{O}^{(0)} \\ &[Q, \mathcal{O}^{(2)}] = d\mathcal{O}^{(1)} && d\mathcal{O}^{(2)} = 0 \end{aligned} \tag{1.8}$$

why is this interesting: consider

$$[Q, \int_\Sigma \mathcal{O}^{(2)}] = \int_\Sigma [Q, \mathcal{O}^{(2)}] = \int_\Sigma d\mathcal{O}^{(1)} = 0 \tag{1.9}$$

which implies $\int_\Sigma \mathcal{O}^{(2)}$ are Q invariant operators. Moreover, if we start with an operator of degree $(1, 1)$, we end up with an operator that has vanishing vector and axial charges, this enables us to freely insert arbitrary many of them, and we will do this thing.

We know that for $2d$ conformal theory, a natural correspondence between operator and state is established, and this also works for superconformal theory. However, we should keep the operator insert on the boundary in R sector and ensure Q as a scalar charge, so we need the topological twist. We do this in the traditional way, by introducing the R symmetry current coupling to the spin connection

$$S \to S + \frac{1}{2}(\int J\bar{\omega} + \bar{J}\omega) \tag{1.10}$$

after the twist, the supercurrent $G^-(z), \bar{G}^-(\bar{z})$ become $(2, 0), (0, 2)$ weight and $\phi_i, \bar{\phi}_i$ become $(0, 0), (1, 1)$ weight. We thus can inserting the field to boundary and get a corresponding state

$$|i\rangle = \phi_i |0\rangle + Q|\cdot\rangle \tag{1.11}$$

remember $Q^\dagger$ becomes a scalar in the same time, we can parametrize the same vacua using anti-chiral field, which is connected to the chiral one by a basis transformation

$$\langle\bar{i}| = \langle j|M^j_{\bar{i}} \tag{1.12}$$

we also has two natural inner products, identifying as the metric on the moduli

$$\eta_{ij} = \langle j|i\rangle \quad g_{i\bar{j}} = \langle\bar{j}|i\rangle \tag{1.13}$$

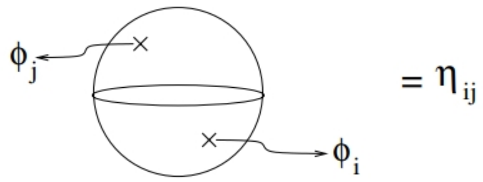


Figure 1.

one of the most important correlation function is the Yukawa coupling

$$\langle\phi_i\phi_j\phi_k\rangle = \langle i|\phi_j|j\rangle = C^l_{jk}\langle i|l\rangle = C^l_{jk}\eta_{il} := C_{jki} \tag{1.14}$$

which implies that C_{ijk} is totally symmetric in indices.

1.1.2 Vacuum Bundle and tt^* equation

Besides the chiral ring itself, we are interested in seeing how the structure of vacua and chiral fields deform as we perturb the theory by marginal chiral fields. We remind ourselves of the vaccum bundle $\{|i(t,\bar{t})\rangle\}$, and study how it varies as a function of moduli parameter. We already know that the chiral ring elements has a dependence on the moduli space, so it can be seen as a bundle over moduli space, where we are able to deduce a covariant derivative with respect to this vector bundle.

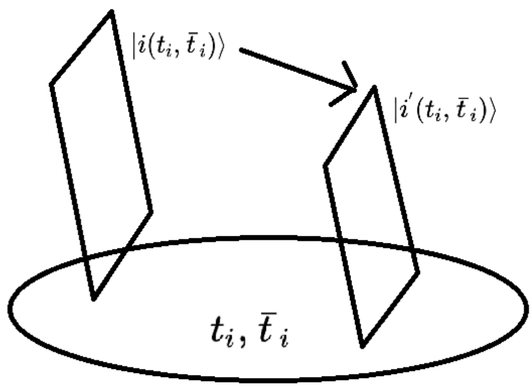


Figure 2.

Using the result of [1](this is also stressed in geometry language in appendix), we know that

$$\bar{\partial}_l C_{ijk} = 0 \tag{1.15}$$

one also introduces a connection on the vacuum bundle

$$D_i|j\rangle := (\partial_i - A_i)|j\rangle \tag{1.16}$$

as well as $D_{\bar{i}}|j\rangle$, there is a interesting matrix relation named tt^* equation , if we view the Yukawa coupling as a matrix element $C_{ij}^k = (C_i)_j^k$

$$\begin{aligned} &[D_i, D_j] = [\bar{D}_i, \bar{D}_j] = [D_i, \bar{C}_j] = [\bar{D}_i, C_j] = 0 \\ &[D_i, C_j] = [D_j, C_i] \;\; [\bar{D}_i, \bar{C}_j] = [\bar{D}_j, \bar{C}_i] \\ &[D_i, \bar{D}_j] = -[C_i, \bar{C}_j] \end{aligned} \tag{1.17}$$

This is the relation between the deduced connection/covariant derivative and the Yukawa coupling.

We have a normalization ambiguity of absorbing fermion zero modes, equally translates to the ambiguity in defining the normalization of the chiral states $|i\rangle$. Consider the line bundle $\mathcal{L}$ over the moduli space generated by the vacuum state $|0\rangle$, two different normalization of path integral give

$$|0\rangle \to f(t^i)|0\rangle \tag{1.18}$$

which can be viewed as a gauge transformation

$$A_{i0}^0 \to A_{i0}^0 + \partial_i f \tag{1.19}$$

And we shall choose the holomorphic gauge

$$A_{i0}^0 = \frac{\partial_i \langle \bar{0}|0\rangle}{\langle \bar{0}|0\rangle} = -\partial_i K \tag{1.20}$$

since $|0\rangle$ is a section of $\mathcal{L}$, Z_0 is a section of of $\mathcal{L}^2$ and Z_g is a section of $\mathcal{L}^{2-2g}$. Where Z_g is the partition function of genus g worldsheet theory,we can see this result in this graph.

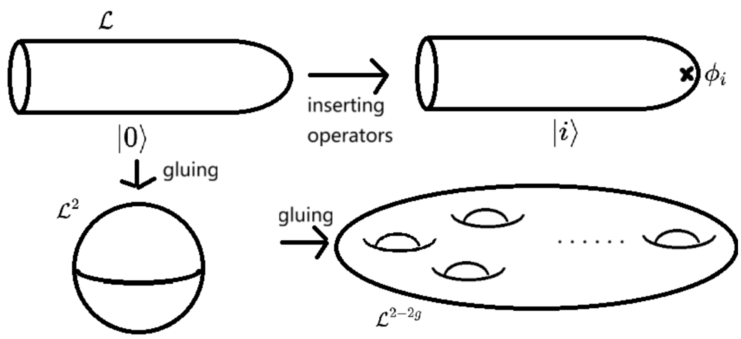


Figure 3.

Together, all physical observables are independent of the choice of normalization ambiguity.

We now leave a short comment on metrics on the moduli space, normally we mean the metric $g_{i\bar{j}}$ defined at the beginning (which is also called the tt^* metric), but some time we also use the so called Zamolodchikov metric, which has close relation to the metric $g_{i\bar{j}}$. The Zamolodchikov metric is defined as a correlation function of marginal operators,

$$G_{i\bar{j}} = \langle \phi_i^{(2)}(1)\bar{\phi}_{\bar{j}}^{(2)}(0)\rangle \tag{1.21}$$

and it is known that this only differs a scaling by Kähler potential with the natural metric we defined already

$$g_{i\bar{j}} = e^{-K} G_{i\bar{j}} \tag{1.22}$$

since naive special geometry definition is $e^{-K} = \langle \bar{0}|0\rangle$,this shows that the so called Zamolodchikov metric is nothing but a the normalized tt^* metric. Zamolodchikov metric is regarded as the Weil-Peterson metric on the moduli space, so in some reference it is also called Weil-Peterson metric.

1.2 Coupling the B model to gravity

There are three steps to be concerned when we coupled a theory with gravity. First we need to write the Lagrangian in covariant form, so covariant derivative and general metric can appear. Next we need to add the dynamical metric term(Einstein-Hilbert) into the total action, therefore graviton can be some dynamical stuff. However, these two steps are not quite hard comparing to the last: we need to do path integral over all possible metrics.

Follow the inspiration from string theory, we first need to regard all metric connected by Weyl(conformal) and Diffimorphism as equivalent one, which generally generate a moduli space of metrics, more over, we are not sensitive with metrics but topology now, so moduli of all possible Riemann surfaces appear.

One may doubt if there is a conformal anomaly that obstruct us, but we can show this can be universally solved by topological twist. Since the original Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{1.23}$$

the nonvanishing central charge results the conformal anomaly.But after the topological twist

$$\tilde{L}_n = L_n - \frac{1}{2}(n+1)J_n \tag{1.24}$$

with the algebra

$$\begin{aligned}
[L_m, J_n] &= -nJ_{m+n} \\
[J_m, J_n] &= \frac{c}{3}m\delta_{m+n,0}
\end{aligned} \tag{1.25}$$

one can show

$$[\tilde{L}_m, \tilde{L}_n] = (m-n)\tilde{L}_{m+n} \tag{1.26}$$

which means the general topological strings can be well defined in any dimensional target space, but we will see why Calabi-Yau threefold is the most important one.

One then borrows the treatment people does in bosonic string theory to B model to couple it with gravity. We can do the exact identify

$$(G^+(z), J(z), T(z), G^-(z)) \leftrightarrow (J_{BRST}(z), - : bc : (z), \{Q_{BRST}, b(z)\}, b(z)) \tag{1.27}$$

and the ghost number of an operator is identified to the axial charge of the B model.We should note the only condition this identifying works is we have an Calabi-Yau threefold, since the virtual dimension of complex structure deformation moduli space of Riemann surface

$$\dim \mathcal{M}_g = h^1(T\Sigma) - h^0(T\Sigma) = -\int_\Sigma ch(T\Sigma) \wedge td(T\Sigma) = 3g - 3 \tag{1.28}$$

matches with the dimension of stable map moduli space

$$c_1(TM) \cdot \beta + \dim M(1-g) \tag{1.29}$$

only at $c_1(TM) = 0$ and $\dim M = 3$, this makes the Calabi-Yau threefold not only important in phenomenological consideration of string theory but mathematically crucial for defining topological strings. Because only if we choose the target space to be Calabi-Yau threefold, we realize mirror symmetry in stringy case (as it confirms the A model), and we are able to put our definition of free energy to higher genus worldsheet.

In bosonic string theory, we use Beltrami differential as basis in the Riemann surface moduli space, we can do similar thing here, we also remark that we omit the measurement

$$\beta^k = \int_{\Sigma_g} d^2 z G^- \mu^k \quad \bar{\beta}^k = \int_{\Sigma_g} d^2 z \bar{G}^- \bar{\mu}^k \tag{1.30}$$

and

$$\mu_g = \left\langle \prod_{k=1}^{3g-3} \beta^k \bar{\beta}^k \right\rangle \tag{1.31}$$

For free energy of the twisted $\mathcal{N} = 2$ theory coupled to gravity at genus $g > 1$, is defined

$$F_g = \int_{\mathcal{M}_g} \mu_g \tag{1.32}$$

The condition for Calabi-Yau threefold is important since only for this condition that $F_g, g > 1$ can be nonzero.Moreover, we defined the prepotential to be genus 0 free energy, and genus 1 free energy is defined by the analogy with bosonic string theory

$$F_1 = \frac{1}{2} \int \frac{d^2\tau}{Im\tau} Tr[(-1)^F F_L F_R q^{H_L} \bar{q}^{H_R}] \tag{1.33}$$

However, although the analogy between B model and bosonic string is convenient, one should not recall it with nostalgia, since G^- cohomology generates a nontrivial chiral ring, but b cohomology in bosonic string theory remains trivial, so we remark only coupling to gravity we can do this analogy.

2 Holomorphic Anomaly and the Recursion Equation

We now go into the topic of holomorphic anomaly equation, we shall first glimpse its appearance in n-pt function and holomorphicity paradox in the first subsection, and derive the holomorphic anomaly equation next. We still follow the original BCOV paper.

2.1 Zamolodchikov derivative and the Holomorphicity paradox

We define the n-point correlation function coupled with gravity, or the topological string n-point function, the worldsheet has a genus of g

$$C^g_{i_1 i_2 \cdots i_n} = \int_{\mathcal{M}_g} \left\langle \int \phi^{(2)}_{i_1} \cdots \int \phi^{(2)}_{i_n} \prod_{k=1}^{3g-3} (\int G^- \mu_k)(\int \bar{G}^- \bar{\mu}_k) \right\rangle \tag{2.1}$$

one might think

$$C^g_{i_1 \cdots i_n} = \partial^n_{i_1 \cdots i_n} F_g \tag{2.2}$$

but this is not true since $C^g_{i_1 \cdots i_n}$ is a section of bundle $Sym^n T \otimes \mathcal{L}^{2-2g}$($T$ is the tangent bundle of the moduli space, for every marginal operator inserted, it is corresponded to a tangent vector, one can refer to figure 2) and F_g is a section of $\mathcal{L}^{2-2g}$, we formally have

$$C^g_{i_1 \cdots i_n} = \mathcal{D}_{i_n} \ldots \mathcal{D}_{i_1} F_g \tag{2.3}$$

In BCOV's second paper, they guessed this $\mathcal{D}_i$ can be replace by the covariant derivative D_i induce by the Zamolodckikov metric, which implies the following recursion relation, we state their result and repeat their argue for the relation in the following part of this subsection, they show that if and only if the formally written covariant derivative is indeed the natural covariant derivative deduced from special geometry, then the following relation will be true

$$e^{2(1-g)K} G^{\bar{j}_1 i_i} \cdots G^{\bar{j}_{n-1} i_{n-1}} C^g_{i_1 \cdots i_{n-1} i_n} = \partial_{i_n} \left(e^{2(1-g)K} G^{\bar{j}_1 i_1} \cdots G^{\bar{j}_{n-1} i_{n-1}} C^g_{i_1 \cdots i_{n-1}} \right) \tag{2.4}$$

We have two things to argue about the naturalness that Zamolodchikov metric arises in the n-point function, which BCOV named as the contact terms argument.

- Firstly, in [2], it is proved in conformal marginal operators, the OPE has the leading term of
$$\phi^{(2)}_i(z)\phi^{(2)}_j(0) \sim \delta^2(z)\Gamma^k_{ij}\phi^{(2)}_k(0) \tag{2.5}$$
where Γ^k_{ij} is the Zamolodchikov metric. BCOV also give a tt^* proof of this conclusion, they conclude the insertion of $\phi^{(2)}_i$ in the correlation is equivalent to
$$\int \phi^{(2)}_i \sim \partial_i - \Gamma_i + (\ldots) \tag{2.6}$$
- Moreover, BCOV also gives a more quantitative result. By calculating the contact term between the additional term of topological twist and the desired operator $\phi^{(2)}_i$, they found the contact term has the form
$$-\partial_i K \frac{R}{2\pi} \tag{2.7}$$

which means for genus g worldsheet

$$\int \phi_i^{(2)} \to \partial_i - \Gamma_i - (2-2g)\partial_i K \tag{2.8}$$

For a specific calculation, a convenient example is the correlations on the sphere. Sphere enjoys a better isometry symmetry so we are able to fix generic three points to $0, 1, \infty$ through $PSL(2, \mathbb{C})$ group.

$$C_{i_1 \cdots i_n} = \left\langle \phi_{i_1}(0)\phi_{i_2}(1)\phi_{i_3}(\infty) \int \phi_{i_4}^{(2)} \cdots \int \phi_{i_n}^{(2)} \right\rangle \tag{2.9}$$

we check if this correlation is holomorphic

$$\begin{aligned}&\bar{\partial}_{\bar{j}} C_{i_1 \cdots i_n} = \\ &\int \sqrt{g} d^2 z \left\langle \phi_{i_1}(0)\phi_{i_2}(1)\phi_{i_3}(\infty) \int \phi_{i_4}^{(2)} \cdots \int \phi_{i_n}^{(2)} \oint_{C_z'} G^+ \oint_{C_z} \bar{G}^+ \bar{\phi}_{\bar{j}}(z) \right\rangle\end{aligned} \tag{2.10}$$

where C_z and $C_z^{'}$ are small contours enclosing the point z. Now we can deform the $C_z^{'}$ countour around the other operator insertions. Since

$$\begin{aligned}&\oint_{C_w} G^+ \phi_i(w) = 0 \\ &\oint_{C_W} G^+ \phi_i^{(2)} = d\phi_i^{(0,1)}\end{aligned} \tag{2.11}$$

then formally we get

$$\begin{aligned}&\bar{\partial}_{\bar{j}} C_{i_1 \cdots i_n} = \\ &-\sum_{k=4}^{n} \int \sqrt{g} d^2 z \left\langle \phi_{i_1}(0)\phi_{i_2}(1)\phi_{i_3}(\infty) \int \phi_{i_4}^{(2)} \cdots \int d\phi_{i_k}^{(0,1)} \cdots \int \phi_{i_n}^{(2)} \oint_{C_z} \bar{G}^+ \bar{\phi}_{\bar{j}}(z) \right\rangle = 0\end{aligned} \tag{2.12}$$

However, we have another approach directly from tt^* equation, which leads to a paradox and finally leads to the Holomorphic anomaly equation. The paradox comes from the commutator $[\bar{\partial}_{\bar{j}}, D_i]$ doesn't vanish, which implies the curvature of the natural Zamolodchikov connection do not vanishes. For instance, we can consider the 4-point function

$$C_{ijkl} = D_l C_{ijk} \tag{2.13}$$

if $\bar{\partial} C_{ijk} = 0$, we have

$$\begin{aligned}\bar{\partial}_{\bar{m}} C_{ijkl} &= \bar{\partial}_{\bar{m}} D_l C_{ijk} = [\bar{\partial}_{\bar{m}}, D_l] C_{ijk} = \\ &= 2G_{l\bar{m}} C_{ijk} - (R^n_{l\bar{m}i} C_{njk} + permutations) \neq 0\end{aligned} \tag{2.14}$$

and this situation is even worse for higher genus conditions. BCOV explains this paradox by arguing that although we can freely insert operators like $\int d\phi_i^{(0,1)}$, these terms do not vanish upon integration over the Riemann surface, because the corresponding integral gets a nontrivial boundary term when the field $\phi_i^{(0,1)}$ approaches a point where some other operator is inserted. To sum up, $\bar{\partial}_{\bar{j}} C_{i_1 \cdots i_n}$ may get a contribution only from the boundary of $\mathcal{M}_{0,n}$, and since this boundary corresponds to two operators colliding, we see that the n-point function may fail to be holomorphic only because of contact terms.

2.2 The Holomorphic Anomaly Equation

We know that $\bar{\partial}_{\bar{j}} C_{i_1 \cdots i_n}$ gets a contribution from the Riemann moduli's boundary, it is not surprising that $\bar{\partial}_{\bar{j}} C^g_{i_1 \cdots i_n}$ also gets a contribution only from the boundary of the moduli space $\mathcal{M}_{g,n}$. Since this case the boundary is more complicated, we eventually get a recursion relation coding how these contributions connects to each other.

2.2.1 BRST invariance violation check

Another understanding is by the violation of BRST invariance, since classically it is sure that partition function and correlation function are holomorphic on the moduli space with respect to BRST invariance.

We examine the violation of $\bar{\partial}_{\bar{i}} F_g = 0$ now, taking derivative of $\bar{t}^{\bar{i}}$ is generated by a insertion of anti-chiral field $\bar{\phi}_{\bar{i}}$

$$
\begin{aligned}
\frac{\partial}{\partial \bar{t}^{\bar{i}}} F_g &= \int_{\mathcal{M}_g} \left\langle \oint_{C_w} G^+ \oint_{C'_w} \bar{G}^+ \bar{\phi}_{\bar{i}}(w) \prod_{k,\bar{k}}^{3g-3} \beta^k \bar{\beta}^{\bar{k}} \right\rangle \cdot [dm \wedge d\bar{m}] \\
&= \int_{\mathcal{M}_g} 4 \sum_{i\bar{i}=1}^{3g-3} \frac{\partial^2}{\partial m_i \partial \bar{m}_i} \left\langle \bar{\phi}_{\bar{i}} \prod_{k \neq i} \beta^k \prod_{\bar{k} \neq \bar{i}} \beta^{\bar{k}} \right\rangle \cdot [dm \wedge d\bar{m}]
\end{aligned} \tag{2.15}
$$

We give a short remark of how this is derived, first using the OPE

$$
G^+(z) G^-(w) \sim \frac{2T(z)}{z-w} + \cdots \quad \rightarrow \quad \oint_{C_w} dz G^+(z) G^-(w) = 2T(w) \tag{2.16}
$$

and then convert T and $\bar{T}$ into derivatives with respect to the moduli $m, \bar{m}$,which comes from a first order deformation of worldsheet metric (after be quotiented by Weyl&Diff):

$$
\int_\Sigma d^2\sigma \sqrt{h} \tilde{\delta} h^{ab} T_{ab} = \int_\Sigma d^z \mu^{(a)z}_{\bar{z}} \delta m^a T_{zz} + \bar{\mu}^{a\bar{z}}_z \delta \bar{m}^a \bar{T}_{zz} \tag{2.17}
$$

after we insert this into

$$
\delta_h \langle \mathcal{O} \rangle_g = \langle \mathcal{O} \int_{\Sigma_g} \sqrt{h} d^2\sigma \delta h^{\mu\nu} T_{\mu\nu} \rangle_g \tag{2.18}
$$

we have

$$
\frac{\partial}{\partial m^a} \mathcal{O}_g = \langle \mathcal{O} \int_\Sigma d^2 z \mu^{a}_{\bar{z}}{}^{z} T_z z \rangle_g := \langle \mathcal{O} T^a \rangle_g \tag{2.19}
$$

2.2.2 Deriving the Holomorphic Anomaly Equation

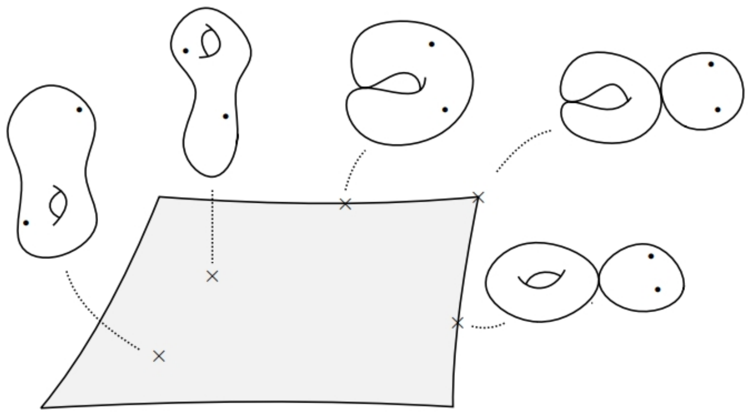


Figure 4.

Back to the theme by viewing the above calculation, BCOV noticed that the integral obtained by $\bar{\partial}_{\bar{i}} F_g$ can be expressed by a integral on the boundary of $\mathcal{M}_g$. The boundary of $\mathcal{M}_g$ consists of $\left[\frac{1}{2}g\right]+1$ irreducible components $\mathcal{D}_g^r$ $r=0,1,\cdots,\left[\frac{1}{2}g\right]$. Surface belonging to $\mathcal{D}_g^0$ are such that they become connected surfaces of genus $(g-1)$ with two punctures upon removal of the nodes.Moreover, $\mathcal{D}_g^r$ consists of surfaces which become upon removal of the nodes, two disconnected surfaces, one of genus r and one of genus $(g-r)$, each with one puncture.We will see them as follow. We remark that in some reference these two classes are named as A and B type sewing, like in Klemm's note, one should not confuse them with A and B model.

We first deal with $\mathcal{D}_g^0$, surface sitting near $\mathcal{D}_g^0$ has a long tube which becomes a node as the surface approaches $\mathcal{D}_g^0$. Thus we can choose coordinates near $\mathcal{D}_g^0$ as (τ, m', z, w) where τ is the moduli of the tube and (m', z, w) are moduli of a genus $(g-1)$ surface with two puncture, where z, w for punctures, m' as ordinary moduli space coordinate. Since the second order derivative of $m, \bar{m}$ in $\bar{\partial}_{\bar{i}} F_g$, at the boundary we will be left with a derivative in the direction normal to $\mathcal{D}_g^0$,which is $\frac{\partial}{\partial Im\tau}$. Together in the limit $\tau \to \infty$, the Beltrami-differentials $\mu^{(z)}$ and $\mu^{(w)}$ associated to the moduli z, w localized near the punctures

$$\int \mu^{(z)} G^- \to \oint_{C_z} G^- \tag{2.20}$$

and those associated to m' reduce to μ' on Σ_{g-1}, the total contribution to $\bar{\partial}_{\bar{i}} F_g$ from $\mathcal{D}_g^0$ is

$$\int_{\mathcal{D}_g^0} \left[dm', dz, dw\right] \frac{\partial}{\partial Im\tau} \left\langle \int_{\Sigma_g} \bar{\phi}_{\bar{i}} \oint_{C_z} G^- \oint_{C_z'} \bar{G}^- \oint_{C_w} G^- \oint_{C_w'} \bar{G}^- \prod_{a=1}^{3g-6} \int_{\Sigma_{g-1}} \mu_a' G^- \int_{\Sigma_{g-1}} \bar{\mu}_a' \bar{G}^- \right\rangle \tag{2.21}$$

The term $\bar{\phi}_{\bar{i}}$ has two probable location, on the tube or outside the tube, as the graph follows. And we will examine the two cases.

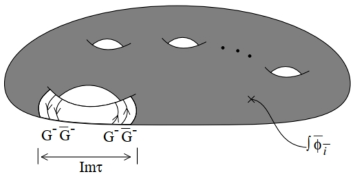


Figure 5.

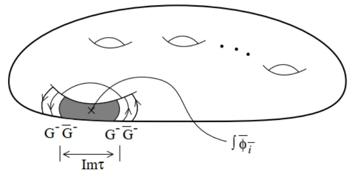


Figure 6.

For the first case, when we approach $\tau \to \infty$ limit, the tube is projected to ground state so the nodes are represented by insertions of $\phi_j(z)$ and $\phi_k(w)$, and the tube is replaced by η^{jk} ,the integrand is

$$\frac{\partial}{\partial Im\tau}\eta^{jk}\left\langle \oint_{C_z} G^- \oint_{C_z'} \bar{G}^-\phi_j(z) \oint_{C_w} G^- \oint_{C_w'} \bar{G}^-\phi_k(w) \int_{\Sigma_{g-1}} \bar{\phi}_{\bar{i}} \prod_{a=1}^{3g-6} \int_{\Sigma_{g-1}} \mu_a' G^- \int_{\Sigma_{g-1}} \bar{\mu}_a' \bar{G}^- \right\rangle \tag{2.22}$$

this turns out to be zero since the latter part do not depend on τ, as defined on Σ_{g-1}. For the second case, the node is represented by an insertion of

$$\phi_j \eta^{jj'} \langle j'| \int \bar{\phi}_{\bar{i}} |k'\rangle \eta^{k'k} \phi_k(w) \tag{2.23}$$

Since

$$\langle j|\bar{\phi}_{\bar{i}}|k\rangle = \langle \bar{j}|\bar{\phi}_{\bar{i}}|\bar{k}\rangle M_j^{\bar{j}} M_k^{\bar{k}} = \bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{\bar{j}j'} G^{\bar{k}k'} \eta_{j'j} \eta_{k'k} \tag{2.24}$$

is independent of the position of $\bar{\phi}_{\bar{i}}$, a trick without rigorous is by approximating the integral result as the volume of the tube, which is $Im\tau$, and the insertion is

$$\phi_j(z)\bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \phi_k(w) \int \mathbf{1} \sim \phi_j(z)\bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \phi_k(w) Im\tau \tag{2.25}$$

this volume cancels with $\frac{\partial}{\partial Im\tau}$, making the integrand

$$\bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \left\langle \oint_{C_z} G^- \oint_{C_z'} \bar{G}^-\phi_j(z) \oint_{C_w} G^- \oint_{C_w'} \bar{G}^-\phi_k(w) \prod_{a=1}^{3g-6} \int_{\Sigma_{g-1}} \mu_a' G^- \int_{\Sigma_{g-1}} \bar{\mu}_a' \bar{G}^- \right\rangle \tag{2.26}$$

the result of the integral is

$$\frac{1}{2}\bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} D_j D_k F_{g-1} \tag{2.27}$$

this is not hard to find out with these two remarks:1.Remember the insertion of $\int \phi^{(2)}$ can be replaced by D_j, 2.The coefficient $\frac{1}{2}$ comes from the symmetry of exchanging z and w. There are some subtleties: as the selection rule comes from zero mode counting affects, we have to restrict

$$q_j + q_k + q_i = \bar{q}_j + \bar{q}_k + \bar{q}_i = 3 \tag{2.28}$$

since $q_i = \bar{q}_i = 1$, the only choice of q_j, q_k that do not annihilated by G^+, G^- is $(q_j, \bar{q}_j) = (q_k, \bar{q}_k) = (1, 1)$, correspond to the marginal deformations of the twisted $\mathcal{N} = 2$ model, which ensures this calculation.

We then deal with $\mathcal{D}_g^r$, recall that a surface in the neighborhood of $\mathcal{D}_g^r$ has a long tube which connects two disconnected surfaces Σ_r and Σ_{g-r}, thus we can choose coordinates near $\mathcal{D}_g^r$ as $(\tau, m^{'}, z, m^{''}, w)$ where τ still characterizes the tube, and $(m^{'}, z) \in \mathcal{M}_{r,1}$, $(m^{''}, w) \in \mathcal{M}_{g-r,1}$. The non-vanishing contribution comes from this graph

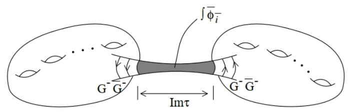


Figure 7.

as

$$\phi_j \eta^{jj^{'}} \langle j^{'} | \bar{\phi}_{\bar{i}} | k^{'} \rangle \eta^{k^{'}k} \phi_k(w) = \bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{\bar{j}j} G^{\bar{k}k} \phi_j(z) \phi_k(w) \tag{2.29}$$

inserting to the integral, the result is

$$\bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{\bar{j}} G^{\bar{k}k} \int_{\mathcal{M}_r} \left\langle \int \phi_j^{(2)} \prod_{a=1}^{3r-3} \int \mu_a^{'} G^{-} \int \bar{\mu}_a^{'} \bar{G}^{-} \right\rangle_{\Sigma_r} \left\langle \int \phi_k^{(2)} \prod_{a=1}^{3(g-r)-3} \int \mu_a^{''} G^{-} \int \bar{\mu}_a^{''} \bar{G}^{-} \right\rangle_{\Sigma_{g-r}} \tag{2.30}$$

perform similar calculation, we obtain

$$\bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{\bar{j}j} G^{\bar{k}k} \phi_j(z) \phi_k(w) D_j F_r D_k F_{g-r} \tag{2.31}$$

We nearly obtain the HAE, with some final remarks: as $r = \frac{1}{2}g$, there is a symmetry between Σ_r and Σ_{g-r}, so a factor $\frac{1}{2}$ is required, for even g, it's

$$\sum_{r=1}^{[\frac{1}{2}g]} \bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{\bar{j}j} G^{\bar{k}k} D_j F_r D_k F_{g-r} \tag{2.32}$$

and for odd g, it's

$$\sum_{r=1}^{\frac{1}{2}g-1} \bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{\bar{j}j} G^{\bar{k}k} D_j F_r D_k F_{g-r} + \frac{1}{2} \sum_{r=1}^{[\frac{1}{2}g]} \bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{\bar{j}j} G^{\bar{k}k} D_j F_{\frac{1}{2}g} D_k F_{\frac{1}{2}g} \tag{2.33}$$

a usual convention is to summarize in a general form

$$\frac{1}{2} \sum_{r=1}^{g-1} \bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{\bar{j}j} G^{\bar{k}k} D_j F_r D_k F_{g-r} \tag{2.34}$$

We combine the above calculation, obtain the Holomorphic Anomaly Equation

$$\bar{\partial}_{\bar{i}} F_g = \frac{1}{2} \bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{\bar{j}j} G^{\bar{k}k} (D_j D_k F_{g-1} + \sum_{r=1}^{g-1} D_j F_r D_k F_{g-r}) \tag{2.35}$$

Last in this subsubsection, we give a short remark on the master equation, which is a general form that combines $g \geq 2$ HAE to one single equation. For summed free energy

$$F = \sum_g F_g g_s^{2g-2} \tag{2.36}$$

If we define

$$\hat{D}_j F = \sum_g g_s^{2g-2} D_j F_g \tag{2.37}$$

then

$$(\bar{\partial}_{\bar{i}} - \bar{\partial}_{\bar{i}} F_1) e^F = \frac{g_s^2}{2} \bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{\bar{j}j} G^{\bar{k}k} \hat{D}_j \hat{D}_k e^F \tag{2.38}$$

is the summing of all genus HAE. We know that e^F has a meaning of partition function, and in some literature, it is also named as "wave function" for string field theory reasons.

2.3 More Remarks About HAE

We shortly glimpse other results BCOV paper covers.

2.3.1 Integrability of HAE

We shortly go over BCOV's result on the integrability of HAE, which is sufficient to prove

$$[d_{\bar{i}}, d_{\bar{j}}] = 0 \tag{2.39}$$

where

$$d_{\bar{i}} = \bar{\partial}_{\bar{i}} - \bar{\partial}_{\bar{i}} F_1 - \frac{g_s^2}{2} \bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{\bar{j}j} G^{\bar{k}k} \hat{D}_j \hat{D}_k \tag{2.40}$$

We can directly see this definition comes from the master equation of HAE, which ensures

$$d_{\bar{i}} e^F \tag{2.41}$$

can be directly integrated. And the proof of the commutator to be zero is even stronger than integrability.BCOV proves this relation by using the following relations

$$\begin{aligned} [\bar{\partial}_{\bar{i}}, D_j]^l_k &= -G_{\bar{i}j} \delta^l_k - G_{\bar{i}k} \delta^l_j + C_{jkm} \bar{C}_{\bar{i}\bar{l}\bar{m}} e^{2K} G^{m\bar{m}} G^{\bar{l}l} \\ \bar{C}_{\bar{i}\bar{j}\bar{k}} = \bar{C}_{\bar{j}\bar{i}\bar{k}} \quad & D_{\bar{i}} \bar{C}_{\bar{j}\bar{k}\bar{l}} = D_{\bar{j}} \bar{C}_{\bar{i}\bar{k}\bar{l}} \quad \partial_i \bar{C}_{\bar{j}\bar{k}\bar{l}} = 0 \end{aligned} \tag{2.42}$$

and by inserting the definition of F_1

$$\partial_i \bar{\partial}_{\bar{j}} F_1 = \frac{1}{2} Tr C_i \bar{C}_{\bar{j}} - \frac{\chi}{24} G_{i\bar{j}} \tag{2.43}$$

after complicate calculations which we omit here, we finally get

$$[d_{\bar{i}}, d_{\bar{j}}] = \frac{g_s^2}{2} \left[\bar{C}_{\bar{i}\bar{k}\bar{l}} e^{2K} G^{\bar{k}k} G^{\bar{l}l} Tr(D_k C_l) \bar{C}_{\bar{j}} \; - \; (\bar{i} \leftrightarrow \bar{j}) \right] = 0 \tag{2.44}$$

2.3.2 HAE of correlation functions

Recall that

$$C^{(g)}_{i_1\cdots i_n} = D_{i_1}\cdots D_{i_n}F_g \tag{2.45}$$

One direct but not simple thought is to generalize Holomorphic Anomaly Equation to correlation functions, and we expect that this generalize form of HAE can come back to the ordinary HAE in a specific limit.

To calculate the desiring $\bar{\partial}_{\bar{i}}C^{(g)}_{i_1\cdots i_n}$, we notice that are two types of contributions. First one is similar to the ordinary HAE, from the boundary of $\mathcal{M}_{g,n}$, the only difference is the moduli of a genus g surface with n punctures. Second one is since many $\bar{\phi}_{\bar{i}}$ is present, we also need to deal with what happen if ϕ_i approach them.

We first consider the first type of contribution, by introducing the result that boundary of $\mathcal{M}_{g,n}$ has two irreducible components $\mathcal{D}^{(0)}_{(g,n)}$ and $\mathcal{D}^{(r,s)}_{(g,n)}$, where r and s represent how many genus and punctures does the boundary components have respectively. From topological considerations, simple properties are

$$\begin{aligned}
&\mathcal{D}^{(0,0)}_{(g,n)} \simeq \mathcal{D}^{(0,1)}_{(g,n)} \simeq \emptyset \\
&\mathcal{D}^{(r,s)}_{(g,n)} \simeq \mathcal{D}^{(g-r,n-s)}_{(g,n)}
\end{aligned} \tag{2.46}$$

Surfaces belonging to $\mathcal{D}^{(0)}_{(g,n)}$ become connected surface of genus $g-1$ with $n+2$ punctures upon removal the node. And surfaces belonging to $\mathcal{D}^{(r,s)}_{(g,n)}$ become two disconnected surfaces, one of genus r with $(s+1)$ punctures, another of genus $(g-r)$ with $(n-s+1)$ punctures. Their graph are as follow respectively

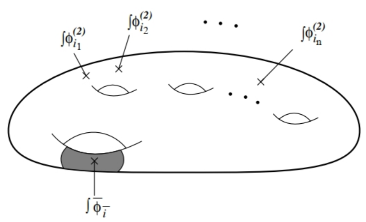


Figure 8.

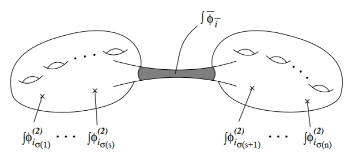


Figure 9.

the total contribution is

$$\frac{1}{2}\bar{C}_{\bar{i}\bar{j}\bar{k}}e^{2K}G^{j\bar{j}}G^{k\bar{k}}C^{(g)}_{jki_1\cdots i_n}+\frac{1}{2}\bar{C}_{\bar{i}\bar{j}\bar{k}}e^{2K}G^{j\bar{j}}G^{k\bar{k}}\sum_{r=0}^{g}\sum_{s=0}^{n}\frac{1}{s!(r-s)!}\sum_{\sigma\in S_n}C^{(r)}_{ji_{\sigma(1)}\cdots i_{\sigma(s)}}C^{(g-r)}_{ji_{\sigma(s+1)}\cdots i_{\sigma(n)}} \tag{2.47}$$

where

$$\begin{aligned}
&C^{(0)}_{i_1\cdots i_n} = D_{i_1}\cdots D_{i_{n-3}}C_{i_{n-2}i_{n-1}i_n} \quad (n\geq 3) \\
&C^{(0)} = C^{(0)}_i = C^{(0)}_{ij} = 0
\end{aligned} \tag{2.48}$$

According to BCOV paper, the second type of contribution is viewed as some curvature singularity, each has the form

$$\pm 2\sum_{s=1}^{n} G_{\bar{i}i_s} C^{(g)}_{i_1\cdots i_{s-1}i_{s+1}\cdots i_n} \tag{2.49}$$

since the integral around the surface has

$$\int R = -2\pi(2g-2+n-1) \tag{2.50}$$

the total HAE for correlation functions is

$$\begin{aligned}
&\bar{\partial}_{\bar{i}} C^{(g)}_{i_1\cdots i_n} = \frac{1}{2}\bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} C^{(g)}_{jki_1\cdots i_n} + \\
&\frac{1}{2}\bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \sum_{r=0}^{g}\sum_{s=0}^{n} \frac{1}{s!(r-s)!} \sum_{\sigma\in S_n} C^{(r)}_{ji_{\sigma(1)}\cdots i_{\sigma(s)}} C^{(g-r)}_{ji_{\sigma(s+1)}\cdots i_{\sigma(n)}} - \\
&- (2g-2+n-1)\sum_{s=1}^{n} G_{\bar{i}i_s} C^{(g)}_{i_1\cdots i_{s-1}i_{s+1}\cdots i_n}
\end{aligned} \tag{2.51}$$

when $n = 0$ this returns to the anomaly equation for free energy. One interesting thing for this HAE is this is also valid for $g = 0$ and $g = 1$ case, which reduces to special geometry relations. This is why HAE is the quantitative characterize of quantum special geometry.

There is also a master equation for correlation function HAE, if we consider the modified partition function (x^is are separate variables, one should not confuse with power)

$$W = \sum_{g=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{n!} g_s^{2g-2} C^{(g)}_{i_1\cdots i_n} x^{i_1}\cdots x^{i_n} + (\frac{\chi}{24} - 1)\ln g_s \tag{2.52}$$

the master equation is

$$\bar{\partial}_{\bar{i}} e^W = \left[\frac{g_s^2}{2}\bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \frac{\partial^2}{\partial x^j \partial x^k} - G_{\bar{i}j} x^j \left(g_s \frac{\partial}{\partial g_s} + x^k \frac{\partial}{\partial x^k}\right)\right] e^W \tag{2.53}$$

3 Integration of Holomorphic Anomaly Equation

BCOV not only gives the HAE itself, but also a method to obtain its solution, and the key object occurs in solving this recursion relation is the Holomorphic Ambiguity.

$$\bar{\partial}_{\bar{i}} A = \bar{\partial}_{\bar{i}} B \Rightarrow A = B + f(z) \tag{3.1}$$

3.1 the Feynman Rules and direct integration

3.1.1 $g = 2$ and $g = 3$

To make it simple, we start by integrating F_2, the genus 2 case reads

$$\bar{\partial}_{\bar{i}} F_2 = \frac{1}{2} \bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \left(D_j \partial_k F_1 + \partial_j F_1 \partial_k F_1 \right) \tag{3.2}$$

Since the Yukawa coupling is totally symmetric in its indices and satisfies

$$D_{\bar{i}} \bar{C}_{\bar{j}\bar{k}\bar{l}} = D_{\bar{j}} \bar{C}_{\bar{i}\bar{k}\bar{l}} \tag{3.3}$$

we can always integrate the Yukawa coupling locally as

$$\bar{C}_{\bar{i}\bar{j}\bar{k}} = e^{-2K} D_{\bar{i}} D_{\bar{j}} \bar{\partial}_{\bar{k}} S \tag{3.4}$$

where S is a local section of $\mathcal{L}^{-2}$, so

$$\bar{C}_{\bar{i}}^{jk} := \bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} = \bar{\partial}_{\bar{i}} S^{jk} \tag{3.5}$$

where $S^{jk} = G^{j\bar{j}} G^{k\bar{k}} \bar{\partial}_{\bar{j}} \bar{\partial}_{\bar{k}} S$. We then write the genus-2 equation using S^{jk}, using the Leibniz rule

$$\bar{\partial}_{\bar{i}} \left[F_2 - \frac{1}{2} S^{jk} (D_j \partial_k F_1 + \partial_j F_1 \partial_k F_1) \right] = -\frac{1}{2} S^{jk} \bar{\partial}_{\bar{i}} (D_j \partial_k F_1 + \partial_j F_1 \partial_k F_1) \tag{3.6}$$

Using the relation of $[\bar{\partial}_{\bar{i}}, D_j]$, we can obtain the right hand part as

$$-\frac{1}{2} \bar{C}_{\bar{i}}^{mn} S^{jk} \left(\frac{1}{2} C_{nmjk} + C_{mnj} \partial_k F_1 + C_{jkm} \partial_n F_1 \right) + \frac{\chi}{24} S_{\bar{i}}^{j} \partial_j F_1 \tag{3.7}$$

using the Leibniz rule again

$$\begin{aligned} &\bar{\partial}_{\bar{i}} \left[F_2 - \frac{1}{2} S^{jk} (D_j \partial_k F_1 + \partial_j F_1 \partial_k F_1) + \frac{1}{4} S^{mn} S^{jk} \left(\frac{1}{2} C_{nmjk} + 2 C_{mnj} \partial_k F_1 \right) - \frac{\chi}{24} S^j \partial_j F_1 \right] \\ &= \frac{1}{4} S^{mn} S^{jk} \bar{\partial}_{\bar{i}} \left(\frac{1}{2} C_{nmjk} + 2 C_{mnj} \partial_k F_1 \right) - \frac{\chi}{24} S^j \bar{\partial}_{\bar{i}} \partial_j F_1 \end{aligned} \tag{3.8}$$

it is observed that the right hand side can be witten in a form of total derivative

$$\begin{aligned} &\frac{1}{4} S^{mn} S^{jk} \bar{\partial}_{\bar{i}} \left(\frac{1}{2} C_{nmjk} + 2 C_{mnj} \partial_k F_1 \right) - \frac{\chi}{24} S^j \bar{\partial}_{\bar{i}} \partial_j F_1 \\ &= \bar{\partial}_{\bar{i}} \left[S^{jk} S^{pq} S^{mn} \left(\frac{1}{8} C_{jkp} C_{mnq} + \frac{1}{12} C_{jpm} C_{kqn} - \frac{\chi}{48} S^j C_{jkl} S^k l + \frac{\chi}{24} (\frac{\chi}{24} - 1) S \right) \right] \end{aligned} \tag{3.9}$$

this means we finally get

$$\bar{\partial}_{\bar{i}} F_2 = \bar{\partial}_{\bar{i}} \left[A\ lot\ of\ terms \right] \tag{3.10}$$

One gets a long but exact expression, up to an ambiguity, of F_2 ($C^{(1)}_{i_1 \cdots i_n}$ means $D_{i_1} \cdots D_{i_n} F_1$)

$$\begin{aligned} F_2 = & \frac{1}{2} S^{ij} C^{(1)}_{ij} + \frac{1}{2} C^{(1)}_i S^{ij} C^{(1)}_j - \frac{1}{8} S^{jk} S^{mn} C_{jkmn} - \frac{1}{2} S^{jk} C_{ijm} S^{mn} C^{(1)}_n + \frac{\chi}{24} S^i C^{(1)}_i \\ & + \frac{1}{8} S^{ij} C_{ijp} S^{pq} C_{qmn} S^{mn} + \frac{1}{12} S^{ij} S^{pq} S^{mn} C_{ipm} C_{jqn} - \frac{\chi}{48} S^i C_{ijk} S^{jk} + \frac{\chi}{24} (\frac{\chi}{24} - 1) S + f_2(t) \end{aligned} \tag{3.11}$$

this expression can be also expressed in Feynman diagram like worldsheet cobinations.

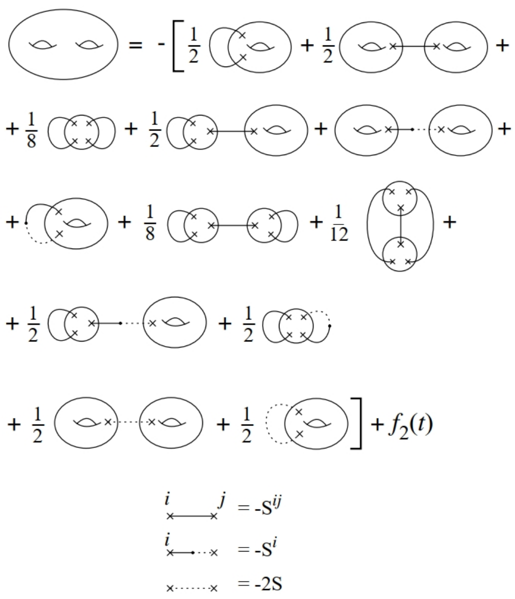


Figure 10.

this method also works for $g = 3$ and even higher, but the expression is so long and I will not type the full expression here, ,by omitting some terms, the Feynman diagram for $g = 3$ is

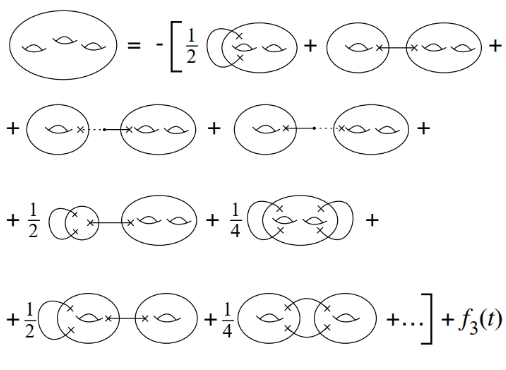


Figure 11.

and the corresponding expression is

$$F_3 = \frac{1}{2}S^{ij}C^{(2)}_{ij} + C^{(1)}_i S^{ij}C^{(2)}_j + (\frac{\chi}{24}+2)S^i C^{(2)}_2 + 2F_2 S^i C^{(1)}_i - \frac{1}{2}S^{ij}C_{ijk}S^{kl}C^{(2)}_l$$
$$-\frac{1}{4}S^{ij}S^{kl}C^{(1)}_{ijkl} - \frac{1}{2}S^{ij}C^{(1)}_{ijk}S^{kl}C^{(1)}_l - \frac{1}{4}S^{ij}S^{kl}C^{(1)}_{jl}C^{(1)}_{jl} + \cdots\cdots\cdots + f_3(t) \quad (3.12)$$

for upper cases, we know we can sum up the graphs according to Feynman rules, we have three types of propagators and many vertices, for propagators

$$K^{ij} = -S^{ij}, \quad K^{i,\varphi} = -S^i, \quad K^{\varphi,\varphi} = -2S \quad (3.13)$$

and many vertices, we generally define as

$$\begin{aligned}
&\tilde{C}^{(g)}_{i_1\cdots i_n,\varphi^{m+1}} = (2g-2+n+m)\tilde{C}^{(g)}_{i_1\cdots i_n,\varphi^m} \\
&\tilde{C}^{(g)}_{i_1\cdots i_n} = C^{(g)}_{i_1\cdots i_n}, \quad \tilde{C}^{(1)}_{\varphi} = \frac{\chi}{24} - 1 \\
&\tilde{C}^{(0)}_{\varphi^m} = 0, \quad \tilde{C}^{(0)}_{i,\varphi^m}, \quad \tilde{C}^{(0)}_{ij,\varphi^m} = 0, \quad \tilde{C}^{(1)} = 0
\end{aligned} \quad (3.14)$$

It's still tolerable for $g=2$ or $g=3$ to write all possible vertices and propagators, but it's quite difficult to generally write all vertices and propagators for higher genus, BCOV developed a generating function of all possible vertices and propagators, by reducing the Feynman rule to the Schwinger-Dyson equation of the finite dimensional system.

3.1.2 arbitrary g

Recalling the master equation for HAE of correlation function,

$$\bar{\partial}_{\bar{i}} e^W = \left[\frac{g_s^2}{2}\bar{C}_{\bar{i}\bar{j}\bar{k}}e^{2K}G^{j\bar{j}}G^{k\bar{k}}\frac{\partial^2}{\partial x^j \partial x^k} - G_{\bar{i}j}x^j\left(g_s\frac{\partial}{\partial g_s} + x^k\frac{\partial}{\partial x^k}\right)\right]e^W \quad (3.15)$$

Beside of the quantity W, BCOV introduces a generating function $\tilde{W}$ of all vertices $\tilde{C}^{(g)}_{i_1\cdots i_n,\varphi^m}$,

$$\tilde{W}(g_s,x,\varphi,t,\bar{t}) = \sum_{g=0}^{\infty}\sum_{n,m=0}^{\infty}\frac{1}{n!m!}g_s^{g-1}\tilde{C}^{(g)}_{i_1\cdots i_n,\varphi^m}x^{i_1}\cdots x^{i_n}\varphi^m \quad (3.16)$$

and there is a relation between $\tilde{W}(g_s,x,\varphi,t,\bar{t})$ and $W(g_s,x,t,\bar{t})$, explicitly,

$$\begin{aligned}
&\tilde{W}(g_s,x,\varphi,t,\bar{t}) \\
&= \sum_{g=0}^{\infty}\sum_{n}^{\infty}\frac{1}{n!}g_s^{2g-2}C^{(g)}_{i_1\cdots i_n}x^{i_1}\cdots x^{i_n}\left(\frac{1}{1-\varphi}\right)^{2g-2+n} + (\frac{\chi}{24}-1)\ln\left(\frac{1}{1-\varphi}\right) \\
&= W\left(\frac{g_s}{1-\varphi},\frac{x}{1-\varphi},t,\bar{t}\right) - \left(\frac{\chi}{24}-1\right)\ln g_s
\end{aligned} \quad (3.17)$$

thus $\tilde{W}$ has a similar master equation

$$\bar{\partial}_{\bar{i}} e^{\tilde{W}} = \left[\frac{g_s^2}{2}\bar{C}^{jk}_{\bar{i}}\frac{\partial^2}{\partial x^j \partial x^k} - G_{\bar{i}j}x^j\frac{\partial}{\partial\varphi}\right]e^{\tilde{W}} \quad (3.18)$$

another similar generating function coding all propagators, where Δ is the inverse of K,defined by

$$
\begin{aligned}
&S^{ij}\Delta_{jk} + S^i\Delta_{k\varphi} = -\delta^i_k \\
&S^{ij}\Delta_{j\varphi} + S^i\Delta_{\varphi\varphi} = 0 \\
&S^i\Delta_{ij} + 2S\Delta_{j\varphi} = 0 \\
&S^i\Delta_{i\varphi} + 2S\Delta_{\varphi\varphi} = -1
\end{aligned} \tag{3.19}
$$

and the generating function is

$$
Y(g_s, x, \varphi, t, \bar{t}) = -\frac{1}{2g_s^2}\left(\Delta_{ij}x^i x^j + 2\Delta_{i\varphi}x^i\varphi + \Delta_{\varphi\varphi}\varphi^2\right) + \frac{1}{2}\ln\left(\frac{\det\Delta}{g_s^2}\right) \tag{3.20}
$$

satisfying

$$
\bar{\partial}_{\bar{i}} e^{\tilde{W}} = \left[-\frac{g_s^2}{2}\bar{C}_{\bar{i}}^{jk}\frac{\partial^2}{\partial x^j\partial x^k} - G_{\bar{i}j}x^j\frac{\partial}{\partial\varphi}\right]e^{\tilde{W}} \tag{3.21}
$$

By considering the integral

$$
Z = \int dxd\varphi exp(Y + \tilde{W}) \tag{3.22}
$$

this integral is regarded as a partition function of a finite dimensional quantum system, and the dynamical degree of freedom are x^i and φ. The perturbative expansion of Z can be obtained by Feynman rules of x^i and φ,

$$
\ln Z = \sum_g g_s^{2g-2}\left[F_g - \frac{1}{2}S^{ij}C^{(g-1)}_{ij} - \frac{1}{2}\sum_{r=1}^{g-1}C^{(r)}_i S^{ij}C^{(g-r)}_j + \cdots\right] \tag{3.23}
$$

another interesting corollary is

$$
\bar{\partial}_{\bar{i}} Z = 0 \tag{3.24}
$$

which indicates that

$$
F_g = -(Feynman\ , nonholomorphic) + holomorphic\ ambuguity \tag{3.25}
$$

is the exact form of all F_g s.

3.1.3 Short Remark on S, for construction of propagators

We can always integrate the Yukawa coupling locally as

$$
\bar{C}_{\bar{i}\bar{j}\bar{k}} = e^{-2K}D_{\bar{i}}D_{\bar{j}}\bar{\partial}_{\bar{k}}S \tag{3.26}
$$

and this S can also be explicitly constructed. Using the definition of Kähler metric and Kähler curvature relation

$$
R^k_{i\bar{j}l} = -\bar{\partial}_{\bar{j}}\Gamma^k_{il} = G_{i\bar{j}}\delta^k_l + G_{k\bar{j}}\delta^k_i - C_{ilm}\bar{C}^{km}_{\bar{j}} \tag{3.27}
$$

the original formula can be rewritten as

$$
\bar{\partial}_{\bar{i}}\left[S^{jk}C_{klm}\right] = \bar{\partial}_{\bar{i}}\left[\partial_l K\delta^j_m + \partial_m K\delta^j_l + \Gamma^j_{lm}\right] \tag{3.28}
$$

this can be easily integrated as

$$S^{ij}C_{jkl} = \delta^i_l \partial_k K + \delta^i_k \partial_k K + \Gamma^i_{kl} + f^i_{kl} \tag{3.29}$$

where f is a meromorphic section of $\mathcal{L}$, which can be expressed as

$$f^i_{kl} = \delta^i_l \partial_k \ln f + \delta^i_k \partial_l \ln f - \sum_{a=1}^{n} v_{l,a} \partial_k v^{i,a} + \tilde{f}^i_{kl} \tag{3.30}$$

$\{v^{i,a}\}_{a=1,\cdots,n}$ are meromorphic tangent vectors which are linearly independent almost everywhere on the moduli space, and $v_{i,a}$ are inverse of $v^{i,a}$, $\tilde{f}$ is a meromorphic section of $T \times Sym^2 T^*$. f and v are generally determined by regularity condition of Kähler potential and metric, as for propagator expression belows, we have invariant combination $e^K|f|^2$ and $G_{1\bar{1}}|v|^2$. We can see this more clear in the quintic case below.

Generally S^{ij} has $\frac{1}{2}n(n+1)$ variables but relation (3.29) has $\frac{1}{2}n^2(n+1)$ constrains, which force people to ensure the choice of $\tilde{f}$ is appropriate.

For case that we only have one modulus, the equation is greatly simplified and the solution is easy to obtain, we can choose $\tilde{f}$ to be 0, the propagators are given as

$$\begin{aligned}
S^{11} &= \frac{1}{C_{111}} \partial \ln \left[2\partial \ln \left(e^K |f|^2 \right) - (G_{1\bar{1}} v)^{-1} \partial (v G_{1\bar{1}}) \right] \\
S^{1} &= \frac{1}{C_{111}} \left[\left(\partial \ln (e^K |f|^2) \right)^2 - v^{-1} \partial (v \partial \ln (e^K |f|^2)) \right] \\
S &= \left[S^1 - \frac{1}{2} D_1 S^{11} - \frac{1}{2} (S^{11})^2 C_{111} \right] \partial \ln (e^K |f|^2) + \frac{1}{2} D_1 S^1 + \frac{1}{2} S^{11} S^1 C_{111}
\end{aligned} \tag{3.31}$$

for general moduli, we have a special solution of S, but actual computation of propagators are extremely miscellaneous

$$S = \frac{1}{2} \left[(n+1) S^i - D_j S^{ij} - S^{ij} S^{kl} C_{jkl} \right] \partial_i \ln (e^K |f|^2) + \frac{1}{2n} \left(D_i S^i + S^i S^{jk} C_{ijk} \right) \tag{3.32}$$

3.2 Explicit Examples

We will give some examples below. In these case, we can see how higher loop partition function is explicitly calculated in a recursion way. We also can have clues that free energy has a close relation to modular forms by doing these calculations. I'm sorry that due to conventions of different models are different, as historical and convenience reasons, that notation of some basic variable may be different in follow subsections, please be careful.

3.2.1 The $\mathbb{Z}_2 \otimes \mathbb{Z}_3$ orbifold

$\mathbb{Z}_2 \otimes \mathbb{Z}_3$ is a result of dividing $T^2 \times T^2 \times T^2$, with each torus having a $\mathbb{Z}_3$ symmetry, by the discrete group generated by

$$g = diag(1, \omega, \omega^2) \quad h = diag(\omega, \omega^2, 1) \tag{3.33}$$

this model has 3 Kähler moduli (every torus has one of them), but no complex moduli, it's a rigid orbifold. The Euler characteristic is $\chi = 168$. The Kähler potential is

$$e^{-K(\tau_a, \bar{\tau}_a)} = i \prod_{a=1}^{3} (\tau_a - \bar{\tau}_a) \tag{3.34}$$

we can compute

$$
\begin{aligned}
G_{a\bar{b}} &= -\frac{\delta_{ab}}{(\tau_a-\bar{\tau}_a)^2} \\
F_1 &= -\kappa\sum_a \ln(\tau_a-\bar{\tau}_a)|\eta^2(\tau_a)|^2
\end{aligned}
\tag{3.35}
$$

where $\kappa = 4$, because of strong symmetry here, we can obtain

$$
\begin{aligned}
S^{ab} &= -\left(\frac{1}{(\tau_c-\bar{\tau}_c)}+2\frac{\eta^{'}(\tau_c)}{\eta(\tau_c)}\right) \\
S^{a} &= \left(\frac{1}{(\tau_b-\bar{\tau}_b)}+2\frac{\eta^{'}(\tau_b)}{\eta(\tau_b)}\right)\left(\frac{1}{(\tau_c-\bar{\tau}_c)}+2\frac{\eta^{'}(\tau_c)}{\eta(\tau_c)}\right) \\
S &= -\prod_a\left(\frac{1}{(\tau_a-\bar{\tau}_a)}+2\frac{\eta^{'}(\tau_a)}{\eta(\tau_a)}\right)
\end{aligned}
\tag{3.36}
$$

we give a short remark of obtaining these propagators, take S^{ab} as example. From equation

$$
\bar{\partial}_c S^{ab} = -\frac{1}{(\tau_c-\bar{\tau}_c)^2} \tag{3.37}
$$

integrating results

$$
S^{ab} = -\frac{1}{(\tau_c-\bar{\tau}_c)}+f(\tau_c) \tag{3.38}
$$

and the condition of modular invaiance fixes $f(\tau) = \frac{2\eta^{'}(\tau)}{\eta(\tau)}$. Using HAE,

$$
\bar{\partial}_a F_2 = -\frac{1}{2}\frac{1}{(\tau a-\bar{\tau}_a)^2}\partial_b F_1\partial_c F_1 \tag{3.39}
$$

and this is integrated

$$
F_2 = \frac{1}{2\kappa}\prod_a \partial_a F_1 = \frac{\kappa^2}{2}\prod_a\left(\frac{1}{(\tau_a-\bar{\tau}_a)}+2\frac{\eta^{'}(\tau_a)}{\eta(\tau_a)}\right) \tag{3.40}
$$

For F_3, a little more complicate. HAE is given

$$
\begin{aligned}
\bar{\partial}_a F_3 = \frac{1}{2}\frac{1}{(\tau_a-\bar{\tau}_a)^2}[&\left(\partial_b+\frac{2}{(\tau_b-\bar{\tau}_b)}\right)\left(\partial_c+\frac{2}{(\tau_c-\bar{\tau}_c)}\right)F_2 \\
&+\partial_b F_1\left(\partial_c+\frac{2}{(\tau_c-\bar{\tau}_c)}\right)F_2+\partial_c F_1\left(\partial_b+\frac{2}{(\tau_b-\bar{\tau}_b)}\right)F_2]
\end{aligned}
\tag{3.41}
$$

we can see that the holomorphic ambiguity has a contribution of modular form of weight 4

$$
\frac{\eta^{''}(\tau)}{\eta(\tau)}-3\left(\frac{\eta^{'}(\tau)}{\eta(\tau)}\right)^2 \tag{3.42}
$$

This phenomenon persists at every genus whenever there is a modular form of appropriate weight. However people do not know the asymptotic behavior of F_g for this model to fix all order ambiguity.

3.2.2 First Glance of Quintic

In every topic of mirror symmetry, Quintic appears, so we are about to discuss it again. We first summarize results of the original BCOV paper, which calculated the free energy of Quintic to $g = 4$. Then I will probably summarize contributions of [3], which calculated the quintic model to $g = 51$, in the next section.

The Quintic has 101 complex moduli and 1 Kähler moduli, the complex moduli can be thought as coefficients of the polynomial and the Kähler moduli can be thought as the Kähler class of $\mathbb{P}^4$, we use its mirror to calculate HAE. Explicitly,

$$W(x_i) = \sum_i x_i^5 - 5\psi x_0 x_1 x_2 x_3 x_4 = 0 \tag{3.43}$$

the holomorphic three form is

$$\Omega = 5\psi \frac{x_4 dx_0 dx_1 dx_2}{\partial W/\partial x_3} \tag{3.44}$$

the Yukawa coupling is

$$C_{\psi\psi\psi} = -\int \Omega \wedge \frac{\partial^3 \Omega}{\partial \psi^3} = \left(\frac{2\pi i}{5}\right)^3 \frac{5\psi^2}{1-\psi^5} \tag{3.45}$$

as $\psi \to 0$ the Kähler potential diverges(in a $|\psi|^2$ behavior) but the metric remains finite, cosidering the invariant combination $e^K|f|^2$ and $G_{\psi\bar{\psi}}|v|^2$, the regularity condition at the origin implies that f should has a zero at $\psi = 0$ while v remains finite,also there is no any additional singularities except $\psi \to \infty$ and $\psi \to 1$ so we have ansatz

$$f(\psi) = \psi(1-\psi^5)^a \quad v(\psi) = (1-\psi^5)^b \tag{3.46}$$

where a and b are some constants. Plug this analysis to 3.31 we know that $\psi \to 0$ the propagators behaves like

$$S^{\psi\psi}\left(\frac{\partial}{\partial\psi}\right)^2 \sim \psi^2\left(\frac{\partial}{\partial\psi}\right)^2 \qquad S^{\psi}\frac{\partial}{\partial\psi} \sim \psi\frac{\partial}{\partial\psi} \qquad S \sim const \tag{3.47}$$

We know that $\psi \to 1$ is the conifold point, a good frame is the canonical coordinate

$$t \sim -\ln(1-\psi^5) \tag{3.48}$$

we can deduce

$$C_{ttt} \sim (1-\psi^5)^2 \tag{3.49}$$

there is a gap condition here

$$F_g \sim \frac{[\partial_t^3 C_{ttt}]^{2g-2}}{[\partial_t C_{ttt}]^{3g-3}} \sim \frac{a_g}{(1-\psi^5)^{2g-2}} \tag{3.50}$$

more carefully analysis gives

$$F_g \sim \frac{A_{2g-2}}{(1-\psi^5)^{2g-2}}\left(\left(\frac{2\pi i}{5}\right)^3 \varpi_0(\psi)\right)^{2g-2} \tag{3.51}$$

where A_n comes from general form of Holomorphic Ambiguity

$$f_g(\psi) = \sum_{g=0}^{2g-2} \frac{A_g}{(1-\psi^5)^{2g-2}} \tag{3.52}$$

We can also find the expression of Kähler potential in the large $\bar{\psi}$ limit, where

$$\begin{aligned} G_{\psi\bar{\psi}} d\bar{\psi} &= C \frac{dt}{d\psi} \frac{d\bar{\psi}}{\bar{\psi}^2} + o(\bar{\psi}^{-3}) \\ K(\psi, \bar{\psi}) &= -\ln \varpi_0(\psi) + o(\bar{\psi}^{-1}) \end{aligned} \tag{3.53}$$

where C is some const and $\varpi_0(\psi)$ is the solution of Picard-Fuchs equation, inserting this results

$$\begin{aligned} S^{\psi\psi} &= \left(\frac{5}{2\pi i}\right)^3 \frac{1-\psi^5}{5\psi^2} \partial_\psi \ln \left(\frac{dt}{d\psi} v \left(\frac{f}{\varpi_0}\right)^2 \right) \\ S^{\psi} &= \left(\frac{5}{2\pi i}\right)^3 \frac{1-\psi^5}{5\psi^2} \left[(\partial_\psi \ln(f/\varpi_0))^2 + v^{-1} \partial_\psi v \partial_\psi \ln(f/\varpi_0) \right] \\ S &= \left[S^\psi - \frac{1}{2} D_\psi S^{\psi\psi} - \frac{1}{2} (S^{\psi\psi})^2 C_{\psi\psi\psi} \right] \partial_\psi \ln(f/\varpi_0) + \frac{1}{2} D_\psi S^\psi + \frac{1}{2} S^{\psi\psi} S^\psi C_{\psi\psi\psi} \end{aligned} \tag{3.54}$$

the genus 2 free energy

$$F_2 = \left(\frac{1}{2} S^{\psi\psi} C^1_{\psi\psi} + \frac{1}{2} C^1_\psi S^{\psi\psi} C^1_\psi - \frac{1}{8} S^{\psi\psi} S^{\psi\psi} C_{\psi\psi\psi\psi} + \cdots \right) + f(\psi) \tag{3.55}$$

where general form of holomorphic ambiguity is given by

$$f(\psi) = A + \frac{B}{(1-\psi^5)} + \frac{C}{(1-\psi^5)^2} \tag{3.56}$$

this reproduce the A model free energy if we transform into canonical coordinate

$$F_2(q) = -\frac{5}{144} + \frac{1}{240} \sum_n^\infty \frac{d_n q^n}{(1-q^n)^2} + \sum_r D_r q^r \tag{3.57}$$

where d_n, D_n has the meaning of Gromov-Witten invariants, counts the number of holomorphic rational curves of degree of degree n and genus 2. The ambiguity constants are fixed by instanton expansion, for genus 2 this is not hard because A model consideration implies there are no genuine genus 2 curve of degree below 3 so contribution from degree below 3 comes entirely from the bubbling of the sphere or a torus. Eventually we have

$$A = -\frac{71375}{288} \quad B = -\frac{10375}{288} \quad C = \frac{625}{48} \tag{3.58}$$

3.2.3 Simple Example From Mirror Curve: Local $\mathbb{P}^2$

In this and next section we actually calculate the refined Free Energy which is a generalized form of free energy, one may first read the contents of refined B model in the next section.

The mirror curve is a expression of toric Calabi-Yau variety, we first recall it quickly. We know that the toric variety is defined by $M = \frac{\mathbb{C}^{k+3} - Z}{G}$, specified by k charge vectors $Q^\alpha \in Z^{k+3}$ satisfying the Calabi-Yau condition. In B model side, we can introduce two $\mathbb{C}$ valued coordinate w^+, w^- as well as homogeneous coordinate $x_i := e^{y_i}$ constrained by

$$(-1)^{Q_0^\alpha} \prod_{i=1}^{k+3} x_i^{Q_i^\alpha} = z_\alpha \tag{3.59}$$

and the local mirror geometry is then defined by

$$w^+ w^- = H = \sum_{i=1}^{k+3} x_i \tag{3.60}$$

makes a conical bundle over a family of Riemann surfaces, and z_αs are complex moduli introduced naturally, the canonical three form is

$$\Omega = \frac{dH dx dy}{Hxy} \tag{3.61}$$

and it was noticed that this has a strong connection with the Seiberg-Witten model, where the theory is also defined by a Riemann surface (the Seiberg-Witten curve). This is quite simple but strong for genus one mirror curve, since it has only one complex moduli, the direct integration is discovered by [4]. And 15 years later the direct integration for genus 2 mirror curve is proposed by [5], which we maybe will see in the next subsubsection.

The direct integration of mirror curves successfully used the properties of modular forms, which some of them we recalled in the appendix. We can obtain the A-period in a similar way as Seiberg-Witten theory

$$\frac{dt}{du} = \sqrt{\frac{E_4(\tau) g_3(u, m_i)}{E_6(\tau) g_2(u, m_i)}} \tag{3.62}$$

and due to the definition of refined free energy, we can compute the free energies at genus one

$$\begin{aligned} F^{(1,0)} &= \frac{1}{24} \ln \left(\Delta \prod_j u^a \prod_j m_j^{b_j} \right) \\ F^{(0,1)} &= \frac{1}{2} \ln \left(\Delta^a \prod_i u_i^a m_j^{b_j} |g_{i\bar{j}}^{-1}| \right) \end{aligned} \tag{3.63}$$

due to the A period we can also calculate the prepotential and the B period

$$\frac{\partial^2 F^{(0,0)}}{\partial t^2} = -\frac{c_0^2}{2\pi i} \tau \tag{3.64}$$

and the propagator is

$$\mathcal{S}^{tt} = \frac{c_0^2}{12} E_2(\tau) \tag{3.65}$$

where c_0 is the intersection number of the A-cycle and the B-cycle.

We can now zoom into the case of local $\mathbb{P}^2$, as the coordinates and constrain is

$$X_0 = uxyw \quad X_1 = x^2 y \quad X_2 = wy^2 \quad X_3 = w^2 x$$
$$z = \frac{X_1 X_2 X_3}{X_0^3} \quad u^{-3} = z \tag{3.66}$$

the mirror curve is given (in Weierstrass form)

$$y^2 = 4x^3 - \frac{1}{12}(1+24z)x - \frac{1}{216}(1+36z+216z^2) \tag{3.67}$$

one calculate the mirror map by the inversion of j function

$$q(z) = -z^3 + 45z^4 - 1512z^5 + 45672z^6 + \cdots \tag{3.68}$$

and A period

$$t = \ln(z) - 6z + 45z^2 - 560z^3 + \cdots \tag{3.69}$$

B period

$$t_D = -\frac{1}{6}(\ln z)^2 + \frac{1}{3}X_A \ln(z) - 3z + \frac{141}{4}z^2 + \frac{1486}{3}z^3 + \cdots \tag{3.70}$$

We can calculate the Yukawa coupling

$$C_{zzz} = -\frac{1}{3}\frac{1}{z^3(1+27z)} \tag{3.71}$$

and the propagator

$$S^{zz} = \frac{3}{4}z^2 + 9z^3 - 54z^4 + 756z^5 + \cdots \tag{3.72}$$

by inputing the prepotential and genus 1 free energy, we can solve the HAE to arbitrary genus, and it is known that the holomorphic ambiguity can be fixed by the gap condition. Which can be calculated explicitly, the first result is given by [6], stating

$$A^{(g)}_{2g-2} = \frac{B_{2g}}{2g(2g-2)} \tag{3.73}$$

3.2.4 Genus 2 mirror curve and the $\mathbb{C}^3/\mathbb{Z}_5$ model

We shortly summarize the direct integration of genus 2 mirror curve's HAE discovered by [5]. We know that it is quite hard to do HAE calculation for general number of complex moduli, since the number of propagators grows factorial. But for cases like mirror curve, we can get strong results from modular forms.

Generally, the mirror curve is by doing Fourier Expansion for Igusa invariants, which is super complicated, at last We can get the τ matrix τ_{ij}, which is used to compute the genus one free energy, which has the same form of genus one case, and the propagators

$$\mathcal{S}^{ij} = \frac{1}{2\pi i}\frac{1}{10}\partial_{\tau_{pq}} \ln(\chi_{10}) C^i_p C^j_q \tag{3.74}$$

From the PF equation, we know the A period, which we can use to compute the prepotential (where C is the intersection matrix)

$$\tau_{ij} = -C_i^k C_j^l \frac{\partial^2 F^{(0,0)}}{\partial t_k \partial t_l} \tag{3.75}$$

and the B period

$$t_D^i = C_j^i \frac{F^{(0,0)}}{\partial t_j} \tag{3.76}$$

We now zoom into the explicit model, the mirror curve takes the form

$$y^2 = -4x^5 + z_1^{-4/5} z_2^{-2/5} x^4 + 2z_1^{-3/5} z_2^{-4/5} x^3 + (1+2z_2) z_1^{-2/5} z_2^{-6/5} x^2 + 2z_1^{-1/5} z_2^{-3/5} x + 1 \tag{3.77}$$

and the igusa invariant is calculated

$$\begin{aligned}
A &= -8z_1^{\frac{2}{5}} z_2^{\frac{4}{5}} \left(-1 + z_2 (4 + 40z_1)\right), \\
B &= 4z_1^{\frac{4}{5}} z_2^{\frac{8}{5}} \left(1 + 24z_1 + 2400z_1^2 z_2^3 - 8z_2 (1 + 25z_1) + z_2^2 \left(16 + 440z_1 - 80z_1^2\right)\right), \\
C &= -8z_1^{\frac{6}{5}} z_2^{\frac{12}{5}} \left(-1 - 20z_1 + 72z_1^2 + 8z_1^2 z_2^4 (1009 + 10900z_1) + 4z_2 \left(3 + 75z_1 + 92z_1^2\right)\right. \\
&\quad \left. -4z_2^2 \left(12 + 365z_1 + 1652z_1^2\right) + 16z_2^3 \left(4 + 145z_1 + 948z_1^2 + 320z_1^3\right)\right), \\
D &= 4096z_1^6 z_2^9 \left(1 + 27z_1 + 3125z_1^2 z_2^3 + 4z_2^2 (4 + 125z_1) - z_2 (8 + 225z_1)\right).
\end{aligned}$$

Figure 12.

using the intersection matrix

$$\begin{pmatrix} -3 & 1 \\ 1 & -2 \end{pmatrix} \tag{3.78}$$

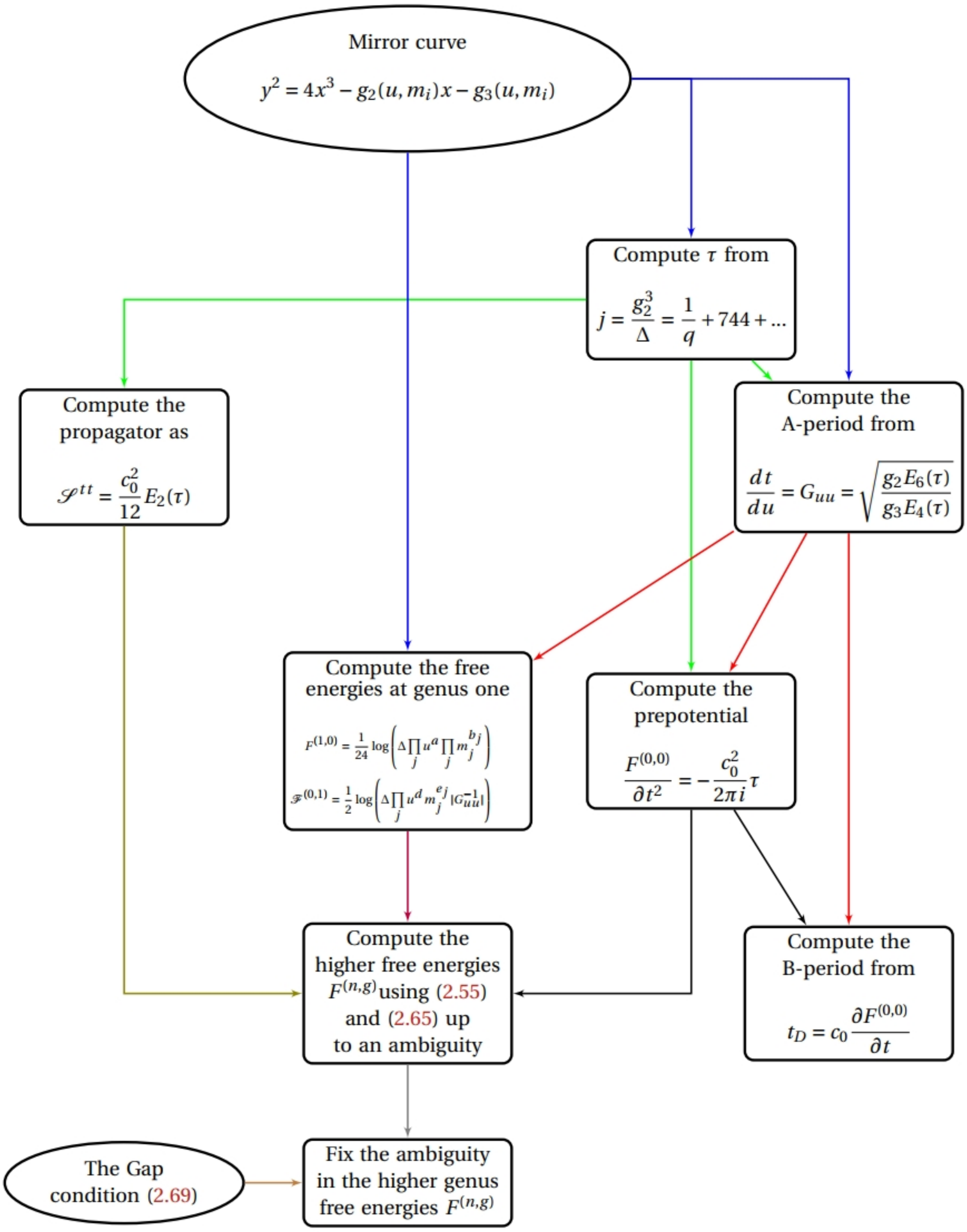
we can get the Yukawa Couplings

$$\begin{aligned}
C_{z_1 z_1 z_1} &= \frac{-2 + 9z_1 - 16z_2 - 95z_1 z_2 + 32z_2^2 + 300z_1 z_2^2}{5z_1^3 \Delta} \\
C_{z_1 z_1 z_2} &= \frac{-1 + 27z_1 - 8z_2 - 210z_1 z_2 + 16z_2^2 + 400z_1 z_2^2}{5z_1^2 z_2 \Delta} \\
C_{z_1 z_2 z_2} &= \frac{-3 + 81z_1 - 14z_2 - 405z_1 z_2 + 8z_2^2 + 325z_1 z_2^2}{5z_1 z_2^2 \Delta} \\
C_{z_2 z_2 z_2} &= \frac{-9 + 243z_1 - 17z_2 - 540z_1 z_2 + 4z_2^2 + 225z_1 z_2^2}{5z_2^3 \Delta}
\end{aligned} \tag{3.79}$$

and the propagators for further integration

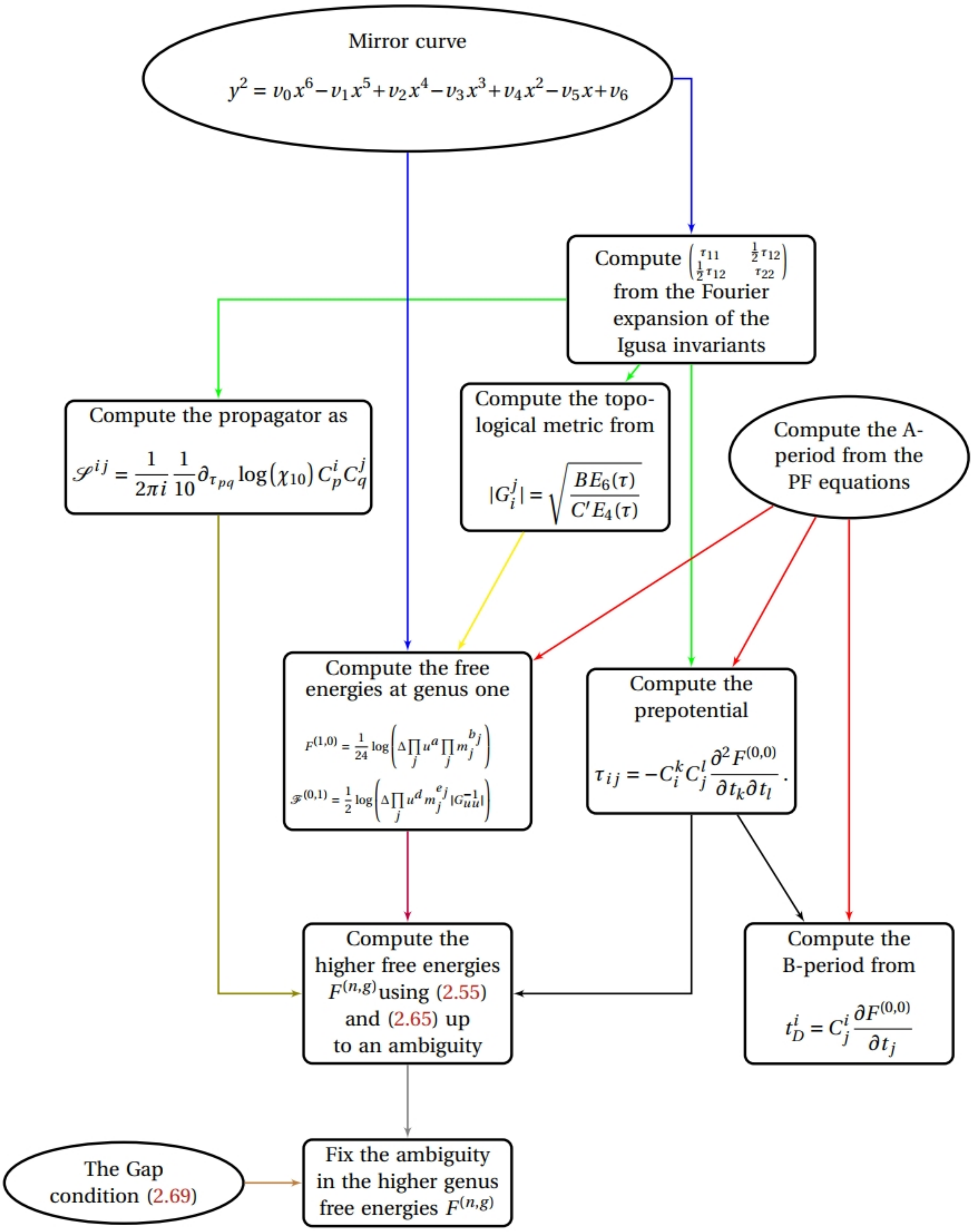
$$\begin{aligned}
S_{z_1 z_1} &= \frac{7}{10} z_1^2 + 9z_1^3 - \frac{3}{10} z_1^2 z_2 - 54z_1^4 - 6z_1^3 z_2 + \cdots \\
S_{z_1 z_2} &= -\frac{3}{20} z_1 z_2 - 3z_1^2 z_2 + \frac{3}{5} z_1 z_2^2 + 18z_1^3 z_2 + 7z_1^2 z_2^2 + \cdots \\
S_{z_2 z_2} &= \frac{3}{10} z_2^2 + z_1 z_2^2 - \frac{6}{5} z_2^3 - 6z_1^2 z_2^2 - 4z_1 z_2^3 + \cdots
\end{aligned} \tag{3.80}$$

the ambiguities are fixed at conifold point $(3, -2/9)$.We end this section by the general procedure of doing direct integral of genus 1 and 2 curve, which are as follow, this graph is from [5], where $(2.55), (2.65)$ are refined HAE.



Mirror curve
$y^2 = 4x^3 - g_2(u, m_i)x - g_3(u, m_i)$
Compute τ from
$j = \frac{g_2^3}{\Delta} = \frac{1}{q} + 744 + \dots$
Compute the propagator as
$\mathscr{S}^{tt} = \frac{c_0^2}{12} E_2(\tau)$
Compute the A-period from
$\frac{dt}{du} = G_{uu} = \sqrt{\frac{g_2 E_6(\tau)}{g_3 E_4(\tau)}}$
Compute the free energies at genus one
$F^{(1,0)} = \frac{1}{24} \log\left(\Delta \prod_j u^a \prod_j m_j^{b_j}\right)$
$\mathscr{F}^{(0,1)} = \frac{1}{2} \log\left(\Delta \prod_j u^d m_j^{e_j} |G_{uu}^{-1}|\right)$
Compute the prepotential
$\frac{F^{(0,0)}}{\partial t^2} = -\frac{c_0^2}{2\pi i} \tau$
Compute the higher free energies $F^{(n,g)}$ using (2.55) and (2.65) up to an ambiguity
Compute the B-period from
$t_D = c_0 \frac{\partial F^{(0,0)}}{\partial t}$
The Gap condition (2.69)
Fix the ambiguity in the higher genus free energies $F^{(n,g)}$

Figure 13.



Mirror curve
$y^2 = v_0 x^6 - v_1 x^5 + v_2 x^4 - v_3 x^3 + v_4 x^2 - v_5 x + v_6$
Compute $\begin{pmatrix} \tau_{11} & \frac{1}{2}\tau_{12} \\ \frac{1}{2}\tau_{12} & \tau_{22} \end{pmatrix}$ from the Fourier expansion of the Igusa invariants
Compute the propagator as
$\mathscr{S}^{ij} = \frac{1}{2\pi i}\frac{1}{10}\partial_{\tau_{pq}} \log(\chi_{10}) C^i_p C^j_q$
Compute the topological metric from
$|G^j_i| = \sqrt{\frac{BE_6(\tau)}{C' E_4(\tau)}}$
Compute the A-period from the PF equations
Compute the free energies at genus one
$F^{(1,0)} = \frac{1}{24}\log\left(\Delta \prod_j u^a \prod_j m_j^{b_j}\right)$
$\mathscr{F}^{(0,1)} = \frac{1}{2}\log\left(\Delta \prod_j u^d m_j^{e_j} |G_{\bar{u}u}^{-1}|\right)$
Compute the prepotential
$\tau_{ij} = -C^k_i C^l_j \frac{\partial^2 F^{(0,0)}}{\partial t_k \partial t_l}.$
Compute the higher free energies $F^{(n,g)}$ using (2.55) and (2.65) up to an ambiguity
Compute the B-period from
$t^i_D = C^i_j \frac{\partial F^{(0,0)}}{\partial t_j}$
The Gap condition (2.69)
Fix the ambiguity in the higher genus free energies $F^{(n,g)}$

Figure 14.

4 Other Aspects of Holomorphic Anomaly Equation

4.1 Discussion of the Holomorphic Ambiguity

4.1.1 General Ansatz

Though direct integration of the Holomorphic Anomaly Equation results a Holomorphic Ambiguity, but this ambiguity is not unconstrained. Since the total $F^{(g)}$ have a quite restricted pole and regularity structure at the critical divisors of $\mathcal{M}_{cs}$, most notably at each conifold divisor there is in the local transversal coordinate t_c a pole of order t_c^{2-2g} and regularity in the sub-leading terms in the $F^{(g)}$. There are also many other types of singularities in generic multi parameter models.

We generally have an ansatz for holomorphic ambiguities in view of the conifold singularities

$$f_g(z) = \sum_{i=1}^{D} \sum_{k=0}^{t(i)} \frac{p_i^{(k)}(z)}{\Delta_i^k} \tag{4.1}$$

where D is the number of components Δ_i of the discriminant and $t(i)$ gives the maximal singularity that one has at the corresponding type of divisor, which is

$$t(i) = 2g - 2 \tag{4.2}$$

for conifold divisors. If in the large complex structure variables the point $z_i \to \infty$ is regular the $p_i^{(k)}(z)$ are polynomials which degrees are bounded by specific models.

4.1.2 Boundary condition from light BPS states

Boundaries in the moduli space correspond to degenerations of the manifold and general properties of the effective action can be inferred from the physics of the lightest states. More precisely the light states relevant to the F_g terms are BPS states. It's convenient to see this in F_1, at the point of maximal unipotent monodromy in the mirror manifold W, the K$\ddot{a}$hler volume of the original manifold is large so the lightest string states are constant maps

$$\Sigma_g \to pt. \in M \tag{4.3}$$

the corresponding F_1 is

$$F_1 = \frac{t_i}{24} \int c_2 \wedge J_i + O(e^{2\pi i t}) \tag{4.4}$$

where J_i is the basis for the K$\ddot{a}$hler cone dual to two cycles. And at the conifold point, W has a nodal singularity with S^3 topology and

$$F_1 = \frac{1}{12} \ln t_D + O(t_D) \tag{4.5}$$

which is physically explained as the effect of integrating out a non-perturbative hypermultiplet namely the extremal black hole with mass$\sim t_D$, whom goes to zero at the conifold and it couples to the $U(1)$ vector in the $\mathcal{N} = 2$ vectormultiplet, whose lowest component is the modulus t_D.

The free energy gives a term

$$S_{1-loop}^{\mathcal{N}=2} = \int d^4 x R_+^2 F(g_s, t) \tag{4.6}$$

in $\mathcal{N} = 2$ supergravity, where R_+ is the self dual part of the curvature. This term is computed by a one-loop integral [9] in a constant graviphoton background, which depends only on the left Lorentz quantum number of BPS particles, which is very similar to the normal Schwinger loop calculation, the latter computes the one-loop effective action in QED, which comes from integrating out massive particles coupling to a constant background photon. We first revisit the QED case, for a self-dual background field $F_{12} = F_{34} = F$, one has

$$S_{1-loop}^{S} = \ln\det(\nabla + m^2 + 2e\sigma_L F) = \int_{\epsilon}^{\infty} \frac{ds}{s} \frac{Tr(-1)^f exp(-sm^2)exp(-2se\sigma_L F)}{4sin^2(seF/2)} \tag{4.7}$$

where $(-1)^f$ depends on the massive particle is boson or fermion, and σ_L is the Cartan element in the left Lorentz representation of the particle.

Then we zoom into the $\mathcal{N} = 2$ SUGRA case, the graviphoton field couples to the mass, the loop has two R_+ and arbitrary graviphoton insertions, only BPS state with the Lorentz quantum number

$$\left[(\frac{1}{2}, 0) + 2(0, 0)\right] \otimes \mathcal{R} \tag{4.8}$$

(for $\mathcal{R}$ an arbitrary representation of $SO(4)$) contributes to the loop. Microscopic BPS states in this loop is related to non-perturbative RR states which are the only charged states in the Type II compactification, comes from ranes wrapping cycles in the Calabi-Yau, and as BPS states their masses are proportional to their central charge. In the IIB picture, it was checked with the beta function in [10] there is precisely one BPS hypermultiplet with the specific Lorentz representation becoming massless at the conifold. In this case the Schwinger-Loop calculation simply becomes

$$F(g_s, t_D) = \int_{\epsilon}^{\infty} \frac{ds}{s} \frac{exp(-st_D)}{4sin^2(sg_s/2)} + O(1) = \sum_{g=2}^{\infty} \left(\frac{g_s}{t_D}\right)^{2g-2} \frac{(-1)^{g-1}B_{2g}}{2g(2g-2)} + O(1) \tag{4.9}$$

which is the gap condition.

4.1.3 Second Glance of Quintic

We can now look how [3] successfully calculated the quintic to $g = 51$ with the holomorphic ambiguity fixed. For simplicity, the convention of quintic is

$$W = \sum_{i}^{5} x_i^5 - 5\psi^{\frac{1}{5}} x_1 x_2 x_3 x_4 x_5 = 0 \tag{4.10}$$

here, after the calculation of Picard-Fuchs equation, we get the prepotential and the B period and the Yukawa coupling

$$C_{\psi\psi\psi} = \frac{\psi^{-1}}{1-\psi} \tag{4.11}$$

at the large radius point, we can solve the Picard-Fuchs equation and use them as basis to treat the prepotential and other symplectic basis results

$$\Pi = \begin{pmatrix} F_0 \\ F_1 \\ X_0 \\ X_1 \end{pmatrix} = \begin{pmatrix} \int_{B_1} \Omega \\ \int_{B_2} \Omega \\ \int_{A^1} \Omega \\ \int_{A^2} \Omega \end{pmatrix} = \omega_0 \begin{pmatrix} 2\mathcal{F}^{(0)} - t\partial_t \mathcal{F}^{(0)} \\ \partial_t \mathcal{F}^{(0)} \\ 1 \\ t \end{pmatrix} = \begin{pmatrix} \omega_3 + c\omega_1 + e\omega_0 \\ -\omega_2 - a\omega_1 + c\omega_0 \\ \omega_0 \\ \omega_1 \end{pmatrix} \tag{4.12}$$

From special geometry we also have the Zamolodichikov metric $G_{\psi\bar{\psi}}$. One key observation of [3], which origins from [11] is to introduce the following variables

$$\begin{aligned} A_p &:= \frac{(\psi\partial_\psi)^p G_{\psi\bar{\psi}}}{G_{\psi\bar{\psi}}} & B_p &:= \frac{(\psi\partial_\psi)^p e^{-K}}{e^{-K}} \\ C &:= C_{\psi\psi\psi}\psi^3 & X &= \frac{1}{1-\psi} \end{aligned} \tag{4.13}$$

and introduce

$$P_g = C^{g-1} F_g \quad P_g^{(n)} = C^{g-1}\psi^n C^{(g)}_{\psi^n} \tag{4.14}$$

a very important result is discovered by defining the variables (u, v_1, v_2, v_3, X) in these implicit equations.

$$\begin{aligned} &B = u \quad A = v_1 - 1 - 2u \quad B_2 = v_2 + uv_1 \\ &B_3 = v_3 - uv_2 + uv_1 X - \frac{2}{5}uX \end{aligned} \tag{4.15}$$

and every P_g is a degree $3g-3$ inhomogenenous polynomial of v_1, v_2, v_3, X, where one assigns the degree $1, 2, 3, 1$ for v_1, v_2, v_3, X respectively, this greatly simplifies the calculation. The HAE is expressed as

$$\begin{aligned} &\frac{\partial P_g}{\partial u} = 0 \\ &\left(\frac{\partial}{\partial v_1} - u\frac{\partial}{\partial v_2} - u(u+X)\frac{\partial}{\partial v_3}\right) P_g = -\frac{1}{2}\left(P_{g-1}^{(2)} + \sum_{r=1}^{g-1} P_r^{(1)} P_{g-r}^{(1)}\right) \end{aligned} \tag{4.16}$$

this also has a holomorphic ambiguity

$$P_g = \mathcal{P}_g(v_1, v_2, v_3, X) + f^{(g)}(X) \tag{4.17}$$

where

$$f^{(g)} = \sum_{i=0}^{3g-3} a_i X^i \tag{4.18}$$

and the $3g-2$ coefficients are fixed by the gap condition around the orbifold point $\psi \to 0$ and the conifold point $\psi \to 1$, the constant term is fixed by the leading coefficients in large complex structure modulus limit $\psi \to \infty$, we use

$$\lim_{t\to\infty} F_{A-model,g} = \frac{(-1)^{g-1} B_{2g} B_{2g-2}}{2g(2g-2)(2g-2)!}\frac{\chi}{2} \tag{4.19}$$

to fix this.So there are still $3g-3$ left for gap condition.

We first see the result around the orbifold point,it is argued that free energy should be analytic at the orbifold point, as there are no massless BPS states so the singularity is not because geometric reasons(so it is not really a ordinary "gap"). Picard-Fuchs equation enjoys 4 different power series solution here

$$\omega_k^{orb} = \psi^{\frac{k}{5}} \sum_{n=0}^{\infty} \frac{\left([\frac{k}{5}]_n\right)^5}{[k]_{5n}} (5^5\psi)^n, \quad k = 1, \cdots, 4 \tag{4.20}$$

then we need to find the basis and Kähler potential besides the orbifold point to define the appropriate coordinate, the method is by doing analytic continuation of the Π and represent it using ω_k^{orb}

$$\begin{pmatrix} F_0 \\ F_1 \\ X_0 \\ X_1 \end{pmatrix} = \psi^{1/5} \frac{\alpha\Gamma^5(\frac{1}{5})}{(2\pi i)^4} \begin{pmatrix} (1-\alpha)(\alpha-1-\alpha^2) \\ \frac{1}{5}(8-3\alpha)(1-\alpha)^2 \\ (1-\alpha+\alpha^2) \\ \frac{1}{5}(1-\alpha)^3 \end{pmatrix} + O(\psi^{2/5}) \tag{4.21}$$

where $\alpha = exp(\frac{2\pi i}{5})$. The appropriate coordinate is

$$s = \frac{\omega_2^{orb}}{\omega_1^{orb}} \tag{4.22}$$

integrating the HAE with the condition and expand the free energies in s, the analytic constrains is equivalent to the regularity of

$$\frac{P_g}{\psi^{\frac{3}{5}(g-1)}} \tag{4.23}$$

which impose

$$\left[\frac{3(g-1)}{5}\right] \tag{4.24}$$

constrains.

About the conifold point, the fixing is direct, also begins with the solution of Picard-Fuchs

$$\Pi_c = \begin{pmatrix} \omega_0^c \\ \omega_1^c \\ \omega_2^c \\ \omega_3^c \end{pmatrix} = \begin{pmatrix} 1 + \frac{2\delta^3}{625} - \frac{83\delta^4}{18750} + \frac{757\delta^5}{156250} + \mathcal{O}\left(\delta^6\right) \\ \delta - \frac{3\delta^2}{10} + \frac{11\delta^3}{25} - \frac{217\delta^4}{2500} + \frac{889\delta^5}{15625} + \mathcal{O}\left(\delta^6\right) \\ \delta^2 - \frac{23\delta^3}{30} + \frac{1049\delta^4}{1800} - \frac{34343\delta^5}{75000} + \mathcal{O}\left(\delta^6\right) \\ \omega_1^c \log(\delta) - \frac{9d^2}{20} - \frac{169d^3}{450} + \frac{27007d^4}{90000} - \frac{152517d^5}{625000} + \mathcal{O}\left(\delta^6\right) \end{pmatrix} \tag{4.25}$$

where $\delta = \psi - 1$ and the appropriate value of coordinate is

$$\hat{t}_D = \frac{\omega_1^c}{\omega_0^c} \tag{4.26}$$

and all free energies can be changed as $\hat{t}_D$ variable functions

$$\begin{aligned}
F^{(0)}_{\text{conf.}} &= -\frac{5}{2}\log\left(\hat{t}_D\right)\hat{t}_D^2 + \frac{5}{12}\left(1-6b_1\right)\hat{t}_D^3 + \left(\frac{5}{12}\left(b_1-3b_2\right) - \frac{89}{1440} - \frac{5}{4}b_1^2\right)\hat{t}_D^4 + \mathcal{O}\left(\hat{t}_D^5\right) \\
F^{(1)}_{\text{conf.}} &= -\frac{\log\left(\hat{t}_D\right)}{12} + \left(\frac{233}{120} - \frac{113b_1}{12}\right)\hat{t}_D + \left(\frac{233b_1}{120} - \frac{113b_1^2}{24} - \frac{107b_2}{12} - \frac{2681}{7200}\right)\hat{t}_D^2 + \mathcal{O}\left(\hat{t}_D^3\right) \\
F^{(2)}_{\text{conf.}} &= \frac{1}{240\hat{t}_D^2} - \left(\frac{120373}{72000} + \frac{11413b_2}{144}\right) + \left(\frac{107369}{150000} - \frac{120373b_1}{36000} + \frac{23533b_2}{720} - \frac{11413b_1b_2}{72}\right)\hat{t}_D + \mathcal{O}\left(\hat{t}_D^2\right)
\end{aligned} \tag{4.27}$$

the gap condition is

$$F^{(g)}_{conifold} \sim \frac{(-1)^{g-1}B_{2g}}{2g(2g-2)\hat{t}_D^{2g-2}} \tag{4.28}$$

this condition fixes $2g-2$ coefficients. However

$$3g-2-(1+\left[\frac{3(g-1)}{5}\right]+(2g-2)) = \left[\frac{2g-2}{5}\right] \tag{4.29}$$

so only $g = 2,3$ this works. How about the other coefficients? It is proved in [9] and [12] that there is a algorithm to calculate the GV invariants directly by cohomologu of the moduli space of $D_2 - D_0$ brane system, this is quite hard so we do not give a summary here. Using this result, [3] gives the calculation to $g = 51$.

4.2 Short review of Refined Holomorphic Anomaly Equation

Inspired by Nekrasov's Instanton Counting Algorithm, people developed Refined Topological string theory. Later, people realized that Holomorphic Anomaly Equation can be also generalized to Refined Topological String, which is introduced in [8][7].

The refined partition function is directly introduce as

$$\ln Z(t,m,\epsilon_1,\epsilon_2) = \sum_{n,g=0}^{\infty}(\epsilon_1+\epsilon_2)^n(\epsilon_1\epsilon_2)^{g-1}F^{(\frac{n}{2},g)}(t,m) \tag{4.30}$$

which is quite similar to the equivariant instanton partition function of $\mathcal{N}=2$ gauge theories, in which t is flat coordinates on the vector multiplet moduli space, m is the bare hyper multiplet masses and ϵ_1, ϵ_2 are the equivariant rotation parameter acting on the so called Ω background, which is parameterized by $s := (\epsilon_1+\epsilon_2)^2$. the refined free energies satisfy for $g_1 + g_2 \geq 2$ a generalized holomorphic anomaly equation

$$\bar{\partial}_{\bar{i}}F^{(g_1,g_2)} = \frac{1}{2}\bar{C}^{jk}_{\bar{i}}\left(D_jD_kF^{(g_1,g_2-1)} + \sum_{r_1,r_2}{}' D_jF^{(r_1,r_2)}D_kF^{(g_1-r_1,g_2-r_2)}\right) \tag{4.31}$$

where the prime denotes that the sum over r_1, r_2 does not include $(r_1,r_2) = (0,0),(g_1,g_2)$. We can observe that this refined HAE reduced to ordinary HAE when $g_1 = 0$. One might be curious that what is the worldsheet description of refined Holomorphic Anomaly Equation, but so far the answer is still unknown, it was conjectured that

$$F^{(n,g)} = \int_{\mathcal{M}_g}\langle\mathcal{O}^n\prod_{k=1}^{3g-3}\beta^k\bar{\beta}^k\rangle_g dm\wedge d\bar{m} \tag{4.32}$$

for some specific operator $\mathcal{O}$ which we still don't know. For some special cases the free energy can be given explicitly, such as

$$
\begin{aligned}
F^{(n+1,0)} &= \langle \phi^{(0)}(0)\phi^{(0)}(1)\phi^{(0)}(\infty)\mathcal{O}^n\rangle_{g=0} \\
F^{(1,0)} &= \frac{1}{24}\ln\left(\Delta \prod_j u^a \prod_j m_j^{b_j}\right) \\
F^{(0,1)} &= \frac{1}{2}\ln\left(\Delta^a \prod_i u_i^a m_j^{b_j} |g_{i\bar{j}}^{-1}|\right)
\end{aligned}
\tag{4.33}
$$

for integration of the refined HAE, people introduce propagators that has a quite simple form

$$
\bar{\partial}_{\bar{i}} F^{(n,g)} = C_{\bar{i}}^{jk} \frac{\partial F^{(n,g)}}{\partial S^{jk}} \tag{4.34}
$$

which implies that $F^{(n,g)}$ is a polynomial of S^{ij} of degree $3(g+n)-3$. The propagators are overdetermined by a series of equations which is determined by special geometry

$$
\begin{aligned}
D_i S^{kl} &= -C_{imn} S^{km} S^{lm} + f_i^{kl} \\
\Gamma_{ij}^k &= -C_{ijl} S^{kl} + \tilde{f}_i^{kl} \\
\partial_i F^{(0,1)} &= \frac{1}{2} C_{ijk} S^{jk} + A_i
\end{aligned}
\tag{4.35}
$$

the gap condition comes from Schwinger Loop computation near the conifold point

$$
\begin{aligned}
&\mathcal{F}(s, g_s, t) = \int_0^\infty \frac{ds}{s} \frac{exp(-st)}{4sinh(s\epsilon_1/2)sinh(s\epsilon_2/2)} + O(1) \\
&= \left[-\frac{1}{12} + \frac{1}{24}(\epsilon_1+\epsilon_2)^2(\epsilon_1\epsilon_2)^{-1}\right]\ln(t) + \frac{1}{\epsilon_1\epsilon_2}\sum_{g=0}^{\infty}\frac{(2g-3)!}{t^{2g-2}}\sum_{m=0}^{g}\hat{B}_{2g}\hat{B}_{2g-2}\epsilon_1^{2g-2m}\epsilon_2^{2m} + \cdots
\end{aligned}
\tag{4.36}
$$

for $\hat{B}_m = \left(\frac{1}{2^{m-1}} - 1\right)\frac{B_m}{m!}$

A Riemann Surface Moduli

We super shortly revise some conclusions about moduli of Riemann Surfaces, for people who almost forget everything about it, like me.

First, definition. $\mathcal{M}_{g,n}$ is a set of isomorphism classes of genus g and n marked points.

$$\mathcal{M}_{g,n} = \{Riemann\ surfaces\ with\ (g, n)\}/iso. \tag{A.1}$$

where the isomorphism is a biholomorphism that maps marked points to marked points.

Shape, Hurwitz's theorem states that the isomorphism group of any Riemann surface satisfying $2g - 2 + n > 0$ is finite, this type of Riemann surfaces is named stable. Almost all of our discussion is in this part, for it actually exclude only 4 possible trivial cases. And all other moduli spaces are connected, smooth, complex orbifold of dimension

$$\dim(\mathcal{M}_{g,n}) = 3g - 3 + n \tag{A.2}$$

and Harer-Zagier find its Euler characteristic number

$$\chi(\mathcal{M}_{g,n}) = (1 - 2g)_{n-1}\zeta(1 - 2g) \tag{A.3}$$

In two dimension, the conformal transformation are equivalent to holomorphic transformations, so a tangent vector of the moduli space is an infinitesimal change of complex structure,which is parameterized by the Beltrami Differential

$$dz \to dz + \epsilon\mu^z_{\bar{z}} d\bar{z} \tag{A.4}$$

B Special Geometry

We start with elements of the theory, which apply to the complex moduli spaces of Calabi-Yau spaces of any dimensions namely the Weil-Petersson metric on the complex moduli space $\mathcal{M}_{cs}$,exists since the Tian-Todorov theorem the moduli space of Calabi-Yau manifolds is unobstructed. The Kähler potential

$$e^{-K} = i^{n^2}\langle\Omega_n, \bar{\Omega}_n\rangle \tag{B.1}$$

We have Griffith transeversality

$$\partial_i\Omega_n = \alpha_i(z)\Omega_n + \chi_i = H^{n,0} \oplus H^{n-1,1} \tag{B.2}$$

With the notation

$$\alpha_i(z) = -K_i = -\partial_i K \tag{B.3}$$

we have results

$$\begin{aligned} &- K_i e^{-K} = \alpha_i e^{-K} \\ &D_i\Omega_n := (\partial_i + K_i)\Omega_n := \chi_i \in H^{n-1,1} \\ &\bar{D}_{\bar{i}}\bar{\Omega}_n := (\bar{\partial}_{\bar{i}} + K_{\bar{i}})\bar{\Omega}_n := \bar{\chi}_i \end{aligned} \tag{B.4}$$

second line as the basis of deformation space of complex structure.We also know there is a gauge transformation for the nonvanishing form

$$\Omega(z) \to e^{f(z)}\Omega(z) \tag{B.5}$$

the Kähler form transforms then in the Kähler line bundle with Kähler transformations

$$K(z,\bar{z}) \to K(z,\bar{z}) - f(z) - \bar{f}(\bar{z}) \tag{B.6}$$

where e^{-K} is a section of $\mathcal{L} \otimes \bar{\mathcal{L}}$,this gauge transform generates a natural connection. We end with the correlation function here is purely holomorphic

$$\langle D_{i_1} \cdots D_{i_r}\Omega_n, \Omega_n\rangle = \langle \partial_{i_1} \cdots \partial_{1_r}\Omega_n, \Omega_n\rangle \tag{B.7}$$

this is because other deformed terms can't survives the matching. So non-stringy cases, we have

$$\bar{\partial}_{\bar{i}} C_{i_1 \cdots i_r} = 0 \tag{B.8}$$

C Modular Forms

Any genus one curve can be represented in Weierstrass normal form

$$y^2 = 4x^3 - g_2(u, m_i)x - g_3(u, m_i) \tag{C.1}$$

where u is the true modulus or the curve which corresponds to the complex structure modulus and m_i denote possible isomonodromic deformations. The coefficents enjoys a rescaling symmetry and exists an r that rescales them to the Eisenstein Series.

$$\begin{aligned}
&g_2 \to r^4 g_2 \quad g_3 \to r^6 g_3 \\
&E_4 = 12 r^4 g_2 \quad E_6 = 216 r^6 g_3 \quad \Delta_{mod} = r^{12}\Delta_d is \\
&\Delta_{mod} = \frac{1}{1728}(E_4^3(\tau) - E_6^3(\tau)) \quad \Delta_{dis} = g_2^3(u, m_i) - 27 g_3^2(u, m_i)
\end{aligned} \tag{C.2}$$

and the associated j function is

$$j = \frac{E_4^3(\tau)}{E_4^3(\tau) - E_6^2(\tau)} = \frac{1}{q} + 744 + 196884q + \cdots \tag{C.3}$$

Any genus two curve can be represented as hyperelliptic curve

$$y^2 = v_0 x^6 - v_1 x^5 + v_2 x^4 - v_3 x^3 + v_4 x^2 - v_5 x + v_6 = \prod_{i=1}^{6}(x - \lambda_i) \tag{C.4}$$

just like the invariants for $g = 1$, we also have invariants for $g = 2$, and I'm not going to type them here, I just paste the graph here.

$$
\begin{aligned}
A =& 6v_3^2 - 16v_2v_4 + 40v_1v_5 - 240v_0v_6,\\
B =& 48v_6v_2^3 + 4v_4^2v_2^2 - 12v_3v_5v_2^2 + 300v_0v_5^2v_2 + 4v_1v_4v_5v_2 - 180v_1v_3v_6v_2 - 504v_0v_4v_6v_2 + 48v_0v_4^3\\
& - 12v_1v_3v_4^2 - 80v_1^2v_5^2 + 1620v_0^2v_6^2 + 36v_1v_3^2v_5 - 180v_0v_3v_4v_5 + 324v_0v_3^2v_6 + 300v_1^2v_4v_6\\
& - 540v_0v_1v_5v_6,\\
C =& -36v_5^2v_2^4 - 160v_4v_6v_2^4 - 24v_4^3v_2^3 - 96v_0v_6^2v_2^3 + 76v_3v_4v_5v_2^3 + 60v_3^2v_6v_2^3 + 616v_1v_5v_6v_2^3\\
& + 8v_3^2v_4^2v_2^2 + 26v_1v_3v_5^2v_2^2 - 640v_0v_4v_5^2v_2^2 - 900v_1^2v_6^2v_2^2 - 24v_3^3v_5v_2^2 + 28v_1v_4^2v_5v_2^2\\
& + 424v_0v_4^2v_6v_2^2 + 492v_1v_3v_4v_6v_2^2 - 876v_0v_3v_5v_6v_2^2 - 160v_0v_4^4v_2 + 76v_1v_3v_4^3v_2\\
& + 1600v_0v_1v_5^3v_2 + 330v_0v_3^2v_5^2v_2 + 64v_1^2v_4v_5^2v_2 + 3060v_0v_1v_3v_6^2v_2 + 20664v_0^2v_4v_6^2v_2\\
& + 492v_0v_3v_4^2v_5v_2 - 238v_1v_3^2v_4v_5v_2 - 198v_1v_3^3v_6v_2 - 640v_1^2v_4^2v_6v_2 - 18600v_0^2v_5^2v_6v_2\\
& - 468v_0v_3^2v_4v_6v_2 - 1860v_1^2v_3v_5v_6v_2 + 3472v_0v_1v_4v_5v_6v_2 - 36v_1^2v_4^4 + 60v_0v_3^2v_4^3 - 320v_1^3v_5^3\\
& + 2250v_0^2v_3v_5^3 - 119880v_0^3v_6^3 - 24v_1v_3^3v_4^2 + 176v_1^2v_3^2v_5^2 - 900v_0^2v_4^2v_5^2 - 1860v_0v_1v_3v_4v_5^2\\
& - 10044v_0^2v_3^2v_6^2 + 2250v_1^3v_3v_6^2 - 18600v_0v_1^2v_4v_6^2 + 59940v_0^2v_1v_5v_6^2 + 72v_1v_3^4v_5 + 616v_0v_1v_4^3v_5\\
& + 26v_1^2v_3v_4^2v_5 - 198v_0v_3^3v_4v_5 + 162v_0v_3^4v_6 - 96v_0^2v_4^3v_6 - 876v_0v_1v_3v_4^2v_6 - 2240v_0v_1^2v_5^2v_6\\
& + 330v_1^2v_3^2v_4v_6 + 1818v_0v_1v_3^2v_5v_6 + 1600v_1^3v_4v_5v_6 + 3060v_0^2v_3v_4v_5v_6,\\
D =& v_0^2\Delta.
\end{aligned}
$$

Figure 15.

which are called Igusa invariants.

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